Last time Pigeon hole principle: if \( m > n \) and 
\[ f : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\} \text{ is any function then} \]
\[ \exists a, b \in \{1, \ldots, m\} \text{ s.t. } a \neq b \text{ but } f(a) = f(b) \]  
\( (f \text{ is not 1-1}). \)

**Consequences**

- If \( f : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\} \) is a bijection then \( n = m \)
- If \( f : \{1, \ldots, m\} \rightarrow A, \ g : \{1, \ldots, m\} \rightarrow A \) are two bijections 
  then \( n = m \). So we can define 
  \[ |A| = \text{the \# of elements of } A \] 
  to be \( n \) (the cardinality of \( A \)).

We can try to extend these ideas of size to infinite sets:

**Definition.** Two sets \( A \) and \( B \) have the same cardinality if 
there is a bijection \( f : A \rightarrow B \). We write \(|A| = |B|\).

**Intuition.** If there is an injection \( f : A \rightarrow B \) then \(|A| \leq |B|\)
- If there is a surjection \( g : A \rightarrow B \) then \(|A| \geq |B|\).
  This is true for finite sets. With infinite sets one should be careful.

First, a short detour.

**Definition.** The power set \( P(A) \) of a set \( A \) is the 
set of all subsets of \( A \):
\[ P(A) = \{ B \mid B \subseteq A \}. \]
\[ \exists A = \emptyset, \ P(A) = 2^{\emptyset}. \]
\[ \exists A = \{1, 2, 3\}, \ P(A) = 2^3 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}. \]

Note: It looks like \(|P(A)| = 2^{|A|}\) and it's true
(See Sally for a proof)
So for a finite \(|A| < |\mathcal{P}(A)|\).

For any \(A\) there is an injective map \(f: A \to \mathcal{P}(A)\):
\[
define f(a) = \{a\} \cup \{a\} \setminus \{\text{set with one element.}\}
\]

So intuition tells us \(|A| \leq |\mathcal{P}(A)|\).

**Theorem (Cantor).** For any set \(A\) there is no bijection from \(A\) to \(\mathcal{P}(A)\).

**Proof** (if \(A = \emptyset\), \(\mathcal{P}(A) = \{\emptyset\}\), a 1-element set. There is no invertible map \(\emptyset \to \emptyset\).

Now suppose \(A \neq \emptyset\) and suppose there is a bijection \(f: A \to \mathcal{P}(A)\).

Then \(f\) is onto. Now consider
\[
S = \{ a \in A \mid a \notin f(a) \}\.
\]

Since \(f\) is onto, \(\exists x \in A\) so that \(f(x) = S\).

If \(x \in S\), then \(x \notin f(x)\). But \(f(x) = S\). Contradiction.

If \(x \notin S\), then \(x \notin f(x)\). \(\Rightarrow x \in S\). Contradiction again.

**Conclusion.** There is no bijection \(f: A \to \mathcal{P}(A)\).

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**Images and preimages of sets**

Let \(f: A \to B\) be a function between two sets.

For \(X \subseteq A\) (i.e. for \(X \in \mathcal{P}(A)\)) we define
\[
f(X) = \{ b \in B \mid b = f(x) \text{ for some } x \in X \}.
\]

The **image** of \(X\) under \(f\).

For \(Y \subseteq B\) (i.e. for \(Y \in \mathcal{P}(B)\)) we define
\[
f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}.
\]

The **preimage** of \(Y\) under \(f\).
\( f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 \)

\[
f([-1,2]) = \{ x^2 \mid x \in [-1,2] \} = [0,4]
\]

\[
f^{-1}(1,4) = \{ x \mid x^2 \in (1,4) \} = \{ x \mid 1 < x^2 < 4 \} = (-2,-1) \cup (1,2).
\]

**Note** (i) \( f^{-1}(Y) \) looks ambiguous if \( f^{-1} \) exists;

Is \( f^{-1}(Y) \) the preimage of \( Y \) under \( f \)
or the image of \( Y \) under \( f^{-1} \).

Fortunately the two are equal.

(ii) For \( y \in Y \), \( f^{-1}(\{y\}) = f^{-1}(y) \), the \( y \)-level set
of \( f \).

(iii) \( f : A \to B \) induces \( f : \mathcal{P}(A) \to \mathcal{P}(B) \)

\[
x \mapsto f(x)
\]

and \( f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A) \)

\[
y \mapsto f^{-1}(y).
\]

(iv) \( f : A \to B \) is onto \( \iff f(A) = B \).

(v) \( f : A \to B \) is injective \( \iff \forall b \in B, \quad |f^{-1}(b)| = 1 \).

\[
\bigcup_{i \in I} A_i = \bigcap_{i \in I} A_i \quad \text{(p27 of Sally)}
\]

Recall if \( A \) is a collection of sets, we defined

\[
\bigcup A = \{ a \mid a \in A \text{ for some } A \in A \}
\]

\[
\bigcap A = \{ a \mid a \in A \text{ for all } A \in A \}.
\]

A family of sets \( \{ A_i \} \) is a function from a set \( I \)
to some collection of sets that assigns to \( i \in I \) the set \( A_i \).

\[
\bigcup_{i \in I} A_i = \bigcup \{ A_i \mid i \in I \} = \{ a \mid a \in A_i \text{ for some } i \in I \}
\]

\[
\bigcap_{i \in I} A_i = \bigcap \{ A_i \mid i \in I \} = \{ a \mid a \in A_i \text{ for all } i \in I \}.
\]
Exercise 1.7.27(i) For any function \( f : A \rightarrow B \) and for any family \( \{A_i\}_{i \in I} \) of subsets of \( A \) (i.e., for any function \( I \rightarrow \mathcal{P}(A), i \mapsto A_i \) )
\[
 f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)
\]

**Solution**
\[
 b \in \bigcup_{i \in I} f(A_i) \iff b \in f(A_j) \text{ for some } j \in I
\]
\[
 \iff b = f(a) \text{ for some } a \in A_j \text{ and some } j \in I
\]
\[
 \iff b = f(a) \text{ for some } a \in \bigcup_{i \in I} A_i
\]
\[
 \iff b \in f(\bigcup_{i \in I} A_i)
\]

Exercise 1.7.27(iv) For any function \( f : A \rightarrow B \) and for any family \( \{B_j\}_{j \in J} \subseteq \mathcal{P}(B) \)
\[
 f^{-1}\left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} f^{-1}(B_j)
\]

**Solution**
\[
 a \in f^{-1}\left(\bigcap_{j \in J} B_j\right) \iff f(a) \in \bigcap_{j \in J} B_j \iff f(a) \in B_j \text{ for all } j \in J
\]
\[
 \iff a \in f^{-1}(B_j) \text{ for all } j \in J
\]
\[
 \iff a \in \bigcap_{j \in J} f^{-1}(B_j)
\]

**Note**
\[
 f\left(\bigcap_{j \in J} A_j\right) \subseteq \bigcap_{j \in J} f(A_j) \text{ but not necessarily equal.}
\]

\[\exists f : (\mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2\]
\[
 A_1 = (-1, 0) \cap (0, 1) = \emptyset
\]

**But**
\[
 f((-1, 0)) = (0, 1) = f((0, 1))
\]

So if we take \( J = \{1, 2\} \), \( A_1 = (-1, 0) \), \( A_2 = (0, 1) \), we have
\[
 f^\prime(A_1 \cap A_2) \not= f(A_1) \cap f(A_2)