The goals of this course is to help you get better at writing proofs and reading proofs.

Since one learns by doing, we’ll do lots of math.

We’ll construct rational numbers out of integers.

We’ll construct real numbers out of rationals.

Along the way we’ll touch on many other topics.

**WARNING** Sally sweeps some things under the rug.

We’ll start by talking about sets.

Informally a set is a collection of (mathematical) objects called elements of the set.

**Notation** We write “\( x \in A \)” to mean:

“\( x \) is an element of the set \( A \)”

We write \( x \notin A \) if \( x \) is not an element of \( A \).

**Examples**

\[ \mathbb{N} = \text{natural numbers} \]

\[ \mathbb{N} = \{ 1, 2, 3, \ldots \} \]

\[ \mathbb{Z} = \text{the set of integers} \]

\[ \mathbb{Z} = \{ 0, 1, -1, 2, -2, \ldots \} \]

\[ \mathbb{Q} = \text{the set of rational numbers} \]

\[ \mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\} \]

\[ \mathbb{R} = \text{the set of real numbers} \]

We’ll show that there is a real number \( x \) so that \( x^2 = 2 \) and \( x > 0 \) (ie \( \sqrt{2} \) exists).

We’ll show \( \sqrt{2} \notin \mathbb{Q} \) (ie. \( \sqrt{2} \) is not rational).
We take the point of view that two sets $A$ and $B$ are equal (we write $A = B$) if and only if (we write $\iff$) they have the same elements:

$$a \in A \iff a \in B.$$ 

**Definition** The empty set, denoted by $\emptyset$, is the set with no elements.

**Definition** A set $A$ is a subset of a set $B$ (we write $A \subseteq B$ or $A \subset B$) if any element of $A$ is also an element of $B$:

$$a \in A \implies a \in B$$

"implies".

**Examples** $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$

**Remark 1.1** For any set $A$, $\emptyset \subseteq A$. Why?

**Remark 1.2** $A = B \iff (A \subseteq B$ and $B \subseteq A)$

**Exercise (a little silly)** For any set $A$, $A \subseteq A$.

**Proof** For any $a \in A$, $a \in A$, $\Box$

**Definition** The union of two sets $A$ and $B$ is the set $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$.

Note here (and in many other instances "or" is not exclusive: "$x \in A \text{ or } x \in B$" means $x \in A$ or $x \in B$ or both.
Ex  \[ A = \{1, 2, 3, 4\}, \quad B = \{2, 4\}\]
\[ A \cup B = \{1, 2, 3, 4\}. \]

**Definition**  The intersection of two sets \( A \) and \( B \) in the set 
\[ A \cap B = \{x \mid x \in A \text{ and } x \in B\}. \]

Ex  \[ \{1, 2, 3, 4\} \cap \{2, 4, 5\} = \{2, 4\} \]
\[ \{1, 2, 3, 4\} \cap \{4, 5, 6\} = \emptyset \]

**Definition**  Two sets \( A \) and \( B \) are **disjoint** (and only if) 
\[ A \cap B = \emptyset. \]
[They have no common elements.]

**Definition**  Suppose \( A \) is a subset of a set \( X \).
**The complement of \( A \) in \( X \)** is the set 
\[ A^c = \{x \in X \mid x \notin A\}. \]

Ex  \[ \mathbb{N} \subset \mathbb{Z}, \quad \mathbb{N}^c = \{n \in \mathbb{Z} \mid n \notin \mathbb{N}\} = \{0, -1, -2, \ldots\} \]

Ex  \[ X = \mathbb{Z}, \quad A = \text{even integers}. \]
\[ A^c = \text{odd integers}. \]

**Example (1.3.7)** in Sally:
**For any three sets** \( A, B, C \)
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

**Proof**  Suppose \( x \in A \cap (B \cup C) \).
Then \( x \in A \) and \( x \in B \cup C \).
Since \( x \in B \cup C \) either \( x \in B \) or \( x \in C \) (or both).

Now if \( x \in B \), then \( x \in A \cap B \) (since \( x \in A \))

if \( x \in C \) then \( x \in A \cap C \) (since \( x \in A \))

\[ \Rightarrow \quad x \in (A \cap B) \cup (A \cap C). \]

Since \( x \) is an arbitrary element of \( A \cap (B \cup C) \) we conclude:

\[ A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C). \]

Conversely, if \( x \in (A \cap B) \cup (A \cap C) \)

then either \( x \in A \cap B \) (and then \( x \in A \) and \( x \in B \))

or \( x \in A \cap C \) (and then \( x \in A \) and \( x \in C \))

In either case, \( x \in A \)

and \((x \in B \) or \( x \in C \))

Hence \( x \in A \) and \( x \in (B \cup C) \)

\[ \Rightarrow \quad x \in A \cap (B \cup C). \]

\[ \Rightarrow \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C). \]

By Remark 1.2

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \]

**Example**

Suppose \( A, B \) are subsets of a set \( X \). Then

\( (A \cup B)^c = A^c \cap B^c. \)

**Proof**

\[ x \in (A \cup B)^c \iff x \in X \text{ and } x \notin A \cup B \]

\[ \iff x \in X \text{ and } (x \notin A \text{ and } x \notin B) \]

\[ \iff (x \in X \text{ and } x \notin A) \text{ and } (x \in X \text{ and } x \notin B) \]

\[ \iff x \in A^c \text{ and } x \in B^c \]

\[ \iff x \in A^c \cap B^c. \]