1. Prove that if \( p \in \mathbb{N} \) is a prime, \( a, b \in \mathbb{Z} \) and \( p | ab \) then either \( p | a \) or \( p | b \).

Hint: Suppose \( p \) does not divide \( a \). Then \( \gcd(p, a) = 1 \) (why?) and therefore there are \( x, y \in \mathbb{Z} \) such that

\[
1 = xp + ya
\]

(why?). Consequently \( b = b \cdot 1 = b \cdot (xp + ya) = \ldots \). Now argue that \( p \) divides \( b \).

2. Prove that for any two complex numbers \( u, v \in \mathbb{C} \)

\[
\bar{u} + v = \bar{\bar{u}} + \bar{v} \quad \text{and} \quad \bar{u} \cdot v = \bar{\bar{u}} \cdot \bar{v}.
\]

3. Suppose \( \{a_n\}, \{b_n\} \subset \mathbb{R} \) are two sequences so that \( a_n < b_n \) for all \( n \) and moreover

\[
[a_{n+1}, b_{n+1}] \subset [a_n, b_n].
\]

Prove that the intersection \( \bigcap_{n=1}^{\infty} [a_n, b_n] \) is nonempty. Give an example to show that \( \bigcap_{n=1}^{\infty} (a_n, b_n) \) may be empty.

4. Suppose a sequence \( \{a_n\} \) of real numbers converges to \( L \).

(a) Prove that any subsequence \( \{a_{n_k}\} \) of \( \{a_n\} \) has to converge and its limit has to be \( L \) as well.

(b) Prove that the sequence \( a_n = \sin(\frac{\pi}{2} n) \) does not converge. Hint: there is more than one way to prove it. One of them uses (a), others don’t.

5. Suppose \( R \) is a commutative ring and \( a \in R \) is an element of this ring. Prove that the set

\[
aR := \{ax \mid x \in R\}
\]

is an ideal in \( R \).

An ideal of the form \( aR \) for some \( a \in R \) is called principal.

6. Prove that any ideal \( I \) in the ring \( \mathbb{Z} \) of integers is principal. That is, prove that there is \( a \in \mathbb{Z} \) so \( I = a\mathbb{Z} \).

Hint: If \( I = \{0\} \), take \( a = \ldots \). Now suppose \( I \neq \{0\} \). Argue first that \( S = \{n \in I \mid n > 0\} \) is non-empty. Next argue that \( I = a\mathbb{Z} \) where \( a = \min S \). You may wish to use the division algorithm for the last argument; it’s very similar to what you did to prove the existence of g.c.d.’s.

7. Let \( R \) be a commutative ring and \( I, J \subset R \) be two ideals. Prove that

\[
I + J = \{x + y \mid x \in I, y \in J\}
\]

is again an ideal in \( R \). \( I + J \) is called the sum of the ideals \( I \) and \( J \).

8. Let \( a, b \in \mathbb{Z} \) be two integers. Then by problem 7 \( I = a\mathbb{Z} + b\mathbb{Z} \) is an ideal in \( \mathbb{Z} \). By problem 6 \( I = d\mathbb{Z} \) for some \( d \in \mathbb{Z} \). Prove that if \( a, b \) are not both zero then \( |d| = \gcd(a, b) \).