Homework 7, Math 347 , Prof. Eugene Lerman
Due Wednesday, October 17, 2018 (in class)

1a Prove that $|a| \leq |a - b| + |b|$ for any $a, b \in \mathbb{R}$. Hint: triangle inequality.

1b Prove that for any $\epsilon > 0$ and any $x \in \mathbb{R}$
   \[ (x < \epsilon \text{ and } -x < \epsilon) \implies |x| < \epsilon. \]

1c Prove that for any $a, b \in \mathbb{R}$
   \[ ||a| - |b|| \leq |a - b|. \]
   Hint: parts (a) and (b) may be useful.

2 In class we defined a sequence $\{a_n\}$ to converge to $L$ if and only if for any $\epsilon > 0$ there is $N \in \mathbb{R}$ so that for any natural number $n$ with $n \geq N$ we have $|a_n - L| < \epsilon$.

   Prove that the definition above is equivalent to: “\(\forall \epsilon > 0 \exists K \in \mathbb{N}\) so that for any $n \in \mathbb{N}$ with $n > K$ we have $|a_n - L| < \epsilon$.”

3 Prove that if $\{a_n\}$ is a sequence of real numbers that converges to $L > 0$ then there is $n_0 \in \mathbb{N}$ so that $a_n > 0$ for all $n \geq n_0$. Hint: let $\epsilon = L/2$...

4 Let $\{a_n\}$ be a constant sequence of real numbers: $a_n = c$ for some $c \in \mathbb{R}$ and all $n$. Prove that $\{a_n\}$ converges to $c$.

5 Suppose $\{a_n\}$ is a sequence of real numbers, $L \in \mathbb{R}$ and $|a_n - L| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}$ converges to $L$.

6 Suppose $A \subset \mathbb{R}$ is bounded above and $L = \sup A$. Prove that there is a sequence $\{a_n\}$ with $a_n \in A$ and $\lim a_n = L$. Hint: argue that for any $n \in \mathbb{N}$ the number $L - \frac{1}{n}$ is not an upper bound of $A$. Therefore there is $a_n \in A$ with $L - \frac{1}{n} < a_n$. Now use problem 5.

7 Recall that a natural number $p > 1$ is prime if (and only if) the only natural numbers that divide it are 1 and $p$.

   a Prove that $p$ is prime if and only if
   \[ \gcd(p, n) = 1 \text{ or } p \]
   for any $n \in \mathbb{Z}$, $n \neq 0$. Hint: this should be easy.

   b Prove that if $p$ is prime, $n \in \mathbb{Z}$ and $p$ does not divide $n$ then there are $a, b \in \mathbb{Z}$ so that $ap + bn = 1$. Hint: Homework 6, problem 5.

   c Prove that if $p$ is prime and $[n] \in \mathbb{Z}_p$ is not $[0]$ then there is $[b] \in \mathbb{Z}_p$ so that
   \[ [b][n] = [1]. \]
   Hint: (b).

   d Prove that if $m > 1$ is a natural number that is not prime then $\mathbb{Z}_m$ has zero divisors and, in particular, cannot be a field.

   e Prove that $\mathbb{Z}_n$ is a field if and only if $n$ is prime.