#1 \( a \in f^{-1}(\bigcup U_i) \Leftrightarrow f(a) \in \bigcup U_i \Leftrightarrow f(a) \in U_j \text{ for some } j \)
\[ \Rightarrow a \in f^{-1}(U_j) \text{ for some } j \Rightarrow a \in \bigcup U_j \text{ for some } j \]
\[ a \in f^{-1}(\bigcap U_i) \Rightarrow f(a) \in \bigcap U_i \Rightarrow f(a) \in U_i \text{ for all } i \Rightarrow a \in f^{-1}(U_i) \text{ for all } i \Rightarrow a \in \bigcap f^{-1}(U_i) \]

#2 Since \( B \subseteq F \) is bounded below, \( \exists L \in F \text{ st. } L \leq b \forall b \in B \Rightarrow -b \geq -L \forall b \in B \Rightarrow -B \) is bounded above. By the least upper bound property of \( F \)
There exists \( \alpha = \sup(-B) \). Then \( \forall b \in B \), \( -b \leq \alpha \). \( \Rightarrow \forall b \in B \)
\[ -\alpha \leq b \Rightarrow -\alpha \text{ is a lower bound of } B \]
If \( L \) is any lower bound of \( B \), then \( -L \) is an upper bound of \(-B\) as we saw above. Since \( \alpha = \sup(-B) \), \( \alpha \leq -L \Rightarrow L \leq -\alpha \).
\[ \Rightarrow -\alpha \text{ is the greatest lower bound of } B \]

#3 (1) Since \( f(x) = f(x) \), \( x \sim x \forall x \in X \).
(2) If \( x \sim y \), \( f(x) = f(y) \). Then \( f(y) = f(x) \Rightarrow y \sim x \).
(3) If \( x \sim y \) and \( y \sim z \), \( f(x) = f(y) \) and \( f(y) = f(z) \Rightarrow f(x) = f(z) \Rightarrow x \sim z \).
\[ \Rightarrow \sim \text{ is an equivalence relation.} \]
- \( [x] = \{ x' \in X \mid x \sim x' \} \)
- \( f([x]) = \{ f(x') \mid x' \in X \} = f(x') \)
- \( f([x]) \) is well-defined.
- If \( f([x]) = f([x']) \) then \( f(x) = f(x') \Rightarrow x \sim x' \Rightarrow [x] = [x'] \Rightarrow f \) is 1-1.

#4 If \( d_1, d_2 \) are two gcd's of \( a \) and \( b \), then \( d_1, d_2 \) (since \( d_2 \) is a greatest common divisor of \( a \) and \( b \) and \( d_1 \mid a \) and \( d_1 \mid b \))
Similarly \( d_2 \mid d_1 \). By \#7 \#5, \( d_1 = \pm d_2 \).
But \( d_1, d_2 \in \mathbb{N} \Rightarrow d_1 = d_2 \).

#5 Consider \( W = \{ xa + yb \mid x, y \in \mathbb{Z}, \ ax + by \geq 0 \} \).
Since \( a, b > 0 \), \( a^2 + b^2 > 0 \) \( \Rightarrow a, a + b, b \in W \) \( \Rightarrow W \neq \emptyset \).

By well-ordering principle, \( d = \min W \) exists. Since \( d \in W \)
\( d = na + mb \) for some \( n, m \in \mathbb{Z} \).

We now argue that \( d \mid a \).

By the division algorithm, \( a = qd + r \) for some \( q, r \in \mathbb{Z} \)
with \( 0 \leq r < d \). Suppose \( r \neq 0 \). Then \( r > 0 \) and
\( r = a - qd = a - q(na + mb) = (1-qn)a + (q-m)b \)

Since we assumed that \( r > 0 \), \( r \in W \). But \( r < d = \min W \)
Contradiction. Therefore \( r = 0 \). \( \Rightarrow d \mid a \).

Similarly \( d \mid b \).

If \( d \mid a \) and \( d \mid b \) then \( d \mid (na + mb) \) \( \) (HW5 #8)
\( \Rightarrow d \mid d \).

Conclusion: \( d = \min W \). \& gcd of \( a \) and \( b \).