Homework 5, Math 347, Prof. Eugene Lerman
Due Friday, October 5, 2018 (in class)

1. Prove that the union of two countable sets is countable.

2. Prove that the set \( \mathbb{Z} \) of integers is countable.

3. Prove that the set \( \mathbb{Q} \) of rational numbers is countable. Hint: prove that \( \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \) is countable.

4. The Cartesian product \( A_1 \times \ldots \times A_n \) of \( n \) sets \( A_1, \ldots, A_n \) is defined to be the set of all \( n \) tuples \((a_1, \ldots, a_n)\) with \( a_i \in A_i \) for all \( i \):

   \[
   A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}
   \]

   Prove that if the sets \( A_1, \ldots, A_n \) are all countable then so is their product \( A_1 \times \cdots \times A_n \).
   Hint: induction on \( n \).

5. Prove that if a set \( A \) is uncountable and \( A \subset B \) then \( B \) is uncountable.

6. Prove that the set \( \text{Map}(\mathbb{N}, \mathbb{N}) \) of all functions from \( \mathbb{N} \) to itself is not countable. Hint: look at the proof that the set \( \mathbb{R} \) of real numbers is not countable.

The next two problems deal with divisibility of integers. Recall that an integer \( a \) divides an integer \( b \) if and only if there is an integer \( q \) with \( b = qa \). We write \( a \mid b \) is \( a \) divides \( b \). In these exercises you may not assume that an integer can be factored into primes.

7. Suppose \( a, b \) are two nonzero integers. Prove that if \( a \mid b \) and \( b \mid a \) then either \( a = b \) or \( a = -b \).

8. Suppose a nonzero integer \( a \) divides \( b \in \mathbb{Z} \) and divides \( c \in \mathbb{Z} \). Prove that \( a \) divides \( nb + mc \) for all \( n, m \in \mathbb{Z} \).