#1

Induction on \( n = |A| \).

If \( |A| = 1 \) then \( A = \{a\} \) for some \( a \). If \( B \subseteq A \) then either \( B = \emptyset \) or \( B = \{a\} \Rightarrow B \) is finite and \( |B| \leq 1 \).

Now suppose \( |A| = k + 1 \) and suppose for any set \( D \) with \( |D| \leq k \), \( C \subseteq D \Rightarrow C \) is finite and \( |C| \leq k \).

Since \( |A| = k + 1 \) there is a bijection \( f : \{1, \ldots, k + 1\} \rightarrow A \).

Suppose \( B \subseteq A \). Then either \( f(k + 1) \in B \) or \( f(k + 1) \notin B \).

If \( f(k + 1) \notin B \) then \( B = D = \{f(1), \ldots, f(k)\} \).

Note: \( g : \{1, \ldots, k\} \rightarrow D, \ g(i) = f(i) \) is a bijection. So by inductive assumption \( B \) is finite and \( |B| \leq k = |D| < k + 1 = |A| \).

If \( f(k + 1) \in B \), \( B \setminus \{f(k + 1)\} \leq \{f(1), \ldots, f(k)\} \).

By inductive assumption \( \exists l \leq k \) and a bijection \( h : \{1, \ldots, l\} \rightarrow B \setminus \{f(k + 1)\} \).

Hence \( \tilde{h} : \{1, \ldots, l, l + 1\} \rightarrow B \), \( \tilde{h}(i) = \begin{cases} f(k + 1) & i = l + 1 \\ h(i) & i \leq l \end{cases} \).

\( \tilde{h} \) is a bijection. \( \Rightarrow B \) is finite and \( |B| = l + 1 \leq k + 1 = |A| \).

Since \( l \leq k \).

#2

Since \( f : B \rightarrow A \) is injective, \( f : B \rightarrow f(B) \) is also a bijection.

By #1, \( f(B) \) is a finite set and \( |f(B)| \leq |A| \).

Since \( |B| = |f(B)| \), \( B \) is a finite set and \( |B| \leq |A| \).

#3

Suppose \( (g \circ f)(a_1) = (g \circ f)(a_2) \) for some \( a_1, a_2 \in A \).

Then \( g(f(a_1)) = g(f(a_2)) \).

Since \( g \) is injective, \( f(a_1) = f(a_2) \).

Since \( f \) is injective, \( a_1 = a_2 \Rightarrow g \circ f \) is injective.

Similarly, given \( c \in C \), \( \exists b \in B \) s.t. \( g(b) = c \).

Since \( g \) is onto. Since \( f \) is onto, \( \exists a \in A \) s.t. \( f(a) = b \Rightarrow (g \circ f)(a) = c \Rightarrow g \circ f \) is onto.
#4 If \( a \in X \cup Y \) then \( a \in X \) or \( a \in Y \)

If \( a \in X \) then \( f(a) \in f(X) \subseteq f(X) \cup f(Y) \)

If \( a \in Y \) then \( f(a) \in f(Y) \subseteq f(X) \cup f(Y) \)

\[ \Rightarrow f(X \cup Y) \subseteq f(X) \cup f(Y). \]

For any sets \( Z, W \subseteq A \) with \( Z \subseteq W \), \( f(Z) \subseteq f(W) \)

\[ \Rightarrow f(X) \subseteq f(X \cup Y) \quad \text{and} \quad f(Y) \subseteq f(X \cup Y) \]

\[ \Rightarrow f(X) \cup f(Y) \subseteq f(X \cup Y). \]

\[ \therefore f(X \cup Y) = f(X) \cup f(Y). \]

#5 Suppose \( x_1, x_2 \in X \) and \( (f|_X)(x_1) = (f|_X)(x_2) \)

Then, by definition of \( f|_X \), \( f(x_1) = f(x_2) \).

\[ \Rightarrow x_1 = x_2 \] since \( f \) is injective.

\[ \Rightarrow f|_X \) is injective.

#6 If \( n=1 \) \( 2^1-1 = 1 \).

Suppose \( 1 + 2 + \ldots + 2^{k+1} = 2^k - 1 \).

Then \( 1 + 2 + \ldots + 2^{k+1} + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1. \)

\[ \therefore 1 + \ldots + 2^n = 2^{n+1} - 1 \] for all \( n \in \mathbb{N}. \)

#7 See last page of lecture 9.