1.5.2 Since additive inverses are unique, it's enough to check that
\((-a) \cdot b + a \cdot b = 0\). And indeed
\((-a) \cdot b + a \cdot b = ((-a)+a) \cdot b = 0 \cdot b = 0\).

1.5.6 (⇒)
\[0 < a \text{ and } a < b \Rightarrow a^2 = a \cdot a < a \cdot b \]  
By transitivity of \(<\),
\[0 < b \text{ and } a < b \Rightarrow a \cdot b < b \cdot b \]

These two facts imply that \(a^2 < b^2\).

(⇐) Suppose \(0 < a, b \text{ and } a^2 < b^2\). If \(a = b\), then \(a^2 = b^2\).

If \(a > b\), then by the argument above \(a^2 > b^2\).

01 now implies that the only possibility left is \(a < b\).

1.5.8 There are two cases: (i) \(a = b\) and (ii) \(a - b \neq 0\).
If \(a = b\), then \(2ab = ab + ab = a^2 + b^2\).
If \(a - b \neq 0\), then Fact 1.5.5(5) implies that
\[0 < (a - b)^2 = a^2 - 2ab + b^2.\]
Add \(2ab\) to both sides. By (03) we get \(2ab < a^2 + b^2\).

1.5.9 There are two directions (1) \& (2) ⇒ 01 - 04
and 01 - 04 ⇒ (1) \& (2).
Suppose \(\exists P \subseteq \mathbb{Z}\) with properties (1) and (2). Define
\(a < b \iff b - a \in P\). Then
01 holds since by (1) either \(b - a \in P\) (and then \(a < b\)) or \(b - a = 0\) (and then \(b = a\)) or \(- (b - a) \in P\). Since
\[-(b - a) = (-b) + ((-a)) = (-b) + a = a - b\]
\(a - b \in P \Rightarrow a < b\).

02 holds: if \(a < b \text{ and } b < c\) then \(b - a, c - b \in P \Rightarrow c - a = (c - b) + (b - a) \in P\), ⇒ \(a < c\).

03: if \(a < b\), then \(P \ni b - a\). But \(b - a = b - a + 0 = b - a + (c + (-c)) = (b + c) + (-1)(a + c) = (b + c) - (a + c)\).
\[ a + c < b + c. \]

**04**  If \( a \leq b \) and \( c \leq d \), then \( b - a, c - d \in P \Rightarrow\]
\[ cb - ca = (c - d)(b - a) \in P \Rightarrow ca < cb. \]

Conversely, given \( c \in \mathbb{Z} \) satisfying \((01)-(04)\) let
\[ P = \{ x \in \mathbb{Z} \mid 0 < x \}. \]

By \(01\), \( \forall \ v \in \mathbb{Z} \) either \(0 \leq \ v \) or \(0 > -\varepsilon \). That is, either \( a \in P \) or \( a = 0 \), or \( 0 < (-a) \) (Fact 1.5.5(01)) which means that \((-a) \in P\). Thus \(01\) holds for this \( P \).

If \( a, b \in P \) then \(0 \leq a, 0 \leq b \). By definition of \( P \). Applying \((03)\)
we get \( b = 0 + b < a + b \). But \(0 \leq b\), so by \(02\) \(0 < a + b\)
\[ = a + b \in P. \]

By Fact 1.5.5(3) we also have \(0 < a \cdot b\).

\(\therefore a \cdot b \in P. \)

Thus \(02\) holds for \( P \) as well.

1.5.12

We note first that \( k, \ell \in \mathbb{N} \)
\[ \binom{k}{\ell} + \binom{k}{\ell - 1} = \frac{k!}{(\ell - 1)! (k - (\ell - 1))!} + \frac{k!}{\ell! (k - \ell)!} = \]
\[ = \frac{k! (\ell - 1 + k - \ell) - k! (\ell - 1)! (k - \ell)! - \ell! (k - \ell)!}{(\ell - 1)! (k - \ell)!} = \frac{k! (\ell - 1)! (k - \ell)!}{\ell! (k + 1 - \ell)!} = \binom{k + 1}{\ell}. \]

Now let \( A = \{ n \in \mathbb{N} \mid (a + b)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} a^\ell b^{n - \ell} \} \)

Then \(1 \in A\) since
\[ (a + b)^1 = a + b = \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0. \]

Suppose \( k \in A \). Then
\[ (a + b)^{k+1} = (a + b)^k (a + b) = \sum_{\ell=0}^{k} \binom{k}{\ell} a^\ell b^{k - \ell} (a + b) \]
\[ = \sum_{\ell=0}^{k} \binom{k}{\ell} a^\ell b^{k - \ell} + \sum_{\ell=0}^{k} \binom{k}{\ell} a^\ell b^{k + 1 - \ell} = \]
\[ = \sum_{\ell=1}^{k+1} \binom{k+1}{\ell-1} a^\ell b^{k+1 - \ell} + \sum_{\ell=0}^{k} \binom{k}{\ell} a^\ell b^{k+1 - \ell} = \]
\[ = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} a^\ell b^{k+1 - \ell}. \]
\[ a^{k+1} + \sum_{l=1}^{k} \left( \binom{k}{l-1} + \binom{k}{l} \right) a^l b^{k+1-l} + b^{k+1} = a^{k+1} + \sum_{l=1}^{k} \binom{k+1}{l} a^l b^{k+1-l} \]

Therefore \( A = X \), i.e., \((a+b)^n = \sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l} \) \( \forall n \in N \).

1.6.14 Suppose a set \( X \) is a union of a collection of subsets of \( X \) that are pairwise disjoint, which we call a partition of \( X \).

Then any \( a \in X \) lies in some \( S \subseteq X \) which belongs to the partition, \( a \in S \) and \( a \in S \Rightarrow a \sim a \).

Suppose \( a \sim b \). This means again that \( \exists S \subseteq X \) in our partition with \( a, b \in S \). But then \( b \sim a \).

Suppose \( a \sim b \) and \( b \sim c \). Then \( \exists S_1, S_2 \subseteq X \) in the partition with \( a, b \in S_1 \) and \( b, c \in S_2 \). Since \( S_1, S_2 \)
belong to a partition, either \( S_1 \neq S_2 \) (and then \( S_1 \cap S_2 = \emptyset \)) or \( S_1 = S_2 \). Since \( b \in S_1 \) and \( b \in S_2 \), \( S_1 \cap S_2 \neq \emptyset \).

Hence \( S_1 = S_2 \). Since \( a \in S_1 \) and \( c \in S_2 \) this says: \( a \sim c \).