1. Prove that for any two sets $A, B$ we have $A \cap B \subseteq A$ and $B \subseteq A \cup B$.

Solution: If $x \in A \cap B$ then $x \in A$ (and $x \in B$).

Since any element of $A \cap B$ is an element of $A$, $A \cap B \subseteq B$.

Similarly, for any $x \in B$, $x \in A \cup B = \{ x \mid x \in A$ or $x \in B \}$.

Hence $B \subseteq A \cup B$.  \textbf{Note:} If $A \cap B = \emptyset$, there is nothing to prove.

$\emptyset \subseteq C$ for any set $C$.

2. (i) Prove that $A \cup (B \cup C) = (A \cup B) \cup C$.

Solution $x \in A \cup (B \cup C) \iff (x \in A) \text{ or } (x \in B \cup C)$

$\iff (x \in A) \text{ or } (x \in B \text{ or } x \in C)$

$\iff x \text{ an element of } A, \text{ of } B \text{ or of } C$

Similarly, $x \in (A \cup B) \cup C \iff x \text{ an element of } A, \text{ of } B \text{ or of } C$.

Since the sets $A \cup (B \cup C)$ and $(A \cup B) \cup C$ have the same elements, the two sets are equal.

(ii) $A \cup \emptyset = A$

If $x \in A \cup \emptyset$, then $(x \in A$ or $x \in \emptyset$). $x \in \emptyset$ is impossible, so $x \in A$. Hence $A \cup \emptyset \subseteq A$. Convervly, $A \subseteq A \cup \emptyset$

by problem 1 above. Hence $A \cup \emptyset = A$.

(iii) Suppose $A \cap B \subseteq X$. Prove that $(A \cap B)^c = A^c \cup B^c$.

Solution: (a) We proved in class that for any $C, D \subseteq X$

$(C \cup D)^c = C^c \cap D^c$

Claim (b) For any subset $Y \subseteq X$, $(Y^c)^c = Y$.

Proof of claim: Note that for any subset $Z$ of $X$

$x \in Z \iff x \notin Z^c$ and $x \notin Z^c \iff x \in Z$.

Now $x \in Y \iff x \notin Y^c \iff x \in (Y^c)^c$, i.e., $Y = (Y^c)^c$.

Now $A^c \cup B^c = (A \cup B)^c = (A^c \cap B^c)^c = (A^c)^c \cap (B^c)^c = (A \cap B)^c$. 
#3 For any set $X$, $R = \{(x, y) \mid x = y\} \subseteq X \times X$ is an equivalence relation.

Solution
(i) For any $x \in X$, $x = x$. $\Rightarrow R$ is reflexive.
(ii) If $x = y$, then $y = x$. $\Rightarrow R$ is symmetric.
(iii) If $x = y$ and $y = z$, then $x = z$. $\Rightarrow R$ is transitive.

$\therefore R$ is an equivalence relation.

#4 $Q = \{(x, y) \in X \times X \mid x \leq y\}$ is reflexive, transitive, but not symmetric. Is $R$ an equivalence relation?

Proof
(i) For any $x \in X$, $x \leq x$. $\Rightarrow R$ is reflexive.
(ii) If $x \leq y$ and $y \leq z$, then $x \leq z$. $\Rightarrow R$ is transitive.

(iii) $(2, 3) \in R$ since $2 < 3$ but $(3, 2) \notin R$ since $3 \neq 2$.

$\Rightarrow R$ is not symmetric.

Since $R$ is not symmetric, $R$ is not an equivalence relation.

#5 $\emptyset \subseteq X \times X$ is not an equivalence relation because if $R$ is not reflexive. In fact, there is no $x \in X$ with $(x, x) \in R$ since $\emptyset$ has no elements.