1. Let $V$ and $W$ be two vector spaces over a field $F$ and $T : V \to W$ a linear map. Prove that if $T$ is a bijection then its inverse $T^{-1} : W \to V$ is linear as well.

Hints: you need to show that $T^{-1}$ preserves addition and multiplication by scalars. To show that $T^{-1}$ preserves addition, consider $w_1, w_2 \in W$. Show that that $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Now compute $T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2)) = ...

2. Let $F$ be a field. Consider the map $T : F[x] \to F[x]$ defined by
   
   $$T(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + na_n x^{n-1}.$$  

Here $2a_2 := a_2 + a_2$, $3a_3 := a_3 + a_3 + a_3$ and so on. (If $F = \mathbb{R}$ then $T$ is the map that sends a polynomial to its derivative).

(a.) Prove that $T$ is linear.

(b.) Is $T$ injective? Prove your answer.

(c.) Prove that for $F = \mathbb{Q}$ the map $T$ is onto. Is $T$ onto if $F = \mathbb{Z}_2$? Prove your answer.

3. Let $F$ be a field, $\alpha \in F$.

(a.) Prove that map $\varphi : F[x] \to F$ given by 
   
   $$\varphi \left( \sum_{i=0}^{n} b_i x^i \right) := \sum_{i=0}^{n} b_i \alpha^i$$

is a homomorphism. The homomorphism $\varphi$ is called the evaluation at $\alpha$. 

(b.) Prove that $\ker \varphi$ is the ideal $\langle x - \alpha \rangle$ consisting of all multiples of the polynomial $x - \alpha$. 

Hint: division algorithm and/or one of its corollaries.

(c.) Prove that $F[x]/\langle x - \alpha \rangle$ is isomorphic to $F$. Hint: 1st isomorphism theorem should make it easy. There are other ways to do it, too.

4. Let $F$ be a field and $I \subset F[x]$ an ideal. Prove that there is a polynomial $f \in F[x]$ so that $I = \langle f \rangle$. That is, prove that all elements of $I$ are multiples of a single polynomial $f$.

Hints: what is $f$ if $I = \{0\}$? Now assume $I \neq \{0\}$. Consider 

$$W = \{ \deg p | p \in I, p \neq 0 \}.$$ 

Argue that $W$ has the smallest element and pick $f \in I$ so that $\deg f = \min W$. Now argue as in the case of ideals in $\mathbb{Z}$.