Homework 11, Math 347, Prof. Eugene Lerman
Due Friday, November 16, 2018 (in class)

1. Let $R$ and $S$ be two rings.

a. Define addition $+$ and multiplication $\cdot$ on the product $R \times S$ “coordinate-wise:"

$$(r, s) + (r', s') := (r + r', s + s') \quad (r, s) \cdot (r', s') := (rr', ss').$$

Check that $R \times S$ with these two operations forms a ring. Prove that if $R$ and $S$ have ones (“units”), then so does $R \times S$.

b. Prove that $I = \{0\} \times S$ is an ideal in $R \times S$ and that $(R \times S)/I$ is isomorphic to $R$.

Hints: (1) since we are not assuming that either $R$ or $S$ are commutative, you need to check that for all $x \in R \times S$ and for all $i \in I$ both $ix$ and $xi$ are in $I$. Alternatively consider the map $pr_1 : R \times S \to R, pr_1(r, s) = r$. What is its kernel?

(2) Show that the map $i : R \to R \times S$, $i(r) = (r, 0)$ is a ring homomorphism. Compose it with $\pi : R \times S \to (R \times S)/I$ and apply the first isomorphism theorem.

c. Prove that the field $\mathbb{C}$ is not isomorphic to the ring $\mathbb{R} \times \mathbb{R}$, where addition and multiplication on $\mathbb{R} \times \mathbb{R}$ is defined as in (a) above.

2. Consider the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients.

a. Show that

$I = \{a_0 + a_1x \cdots + a_nx^n | n \geq 0, a_0 \text{ even}\}$

is an ideal in $\mathbb{Z}[X]$.

b. Prove that $\mathbb{Z}[X]/I$ is isomorphic to $\mathbb{Z}_2$.

Hint: show that the composition of $i : \mathbb{Z} \to \mathbb{Z}[X], i(a) = a$ and $\pi : \mathbb{Z}[x] \to \mathbb{Z}[x]/I$ is onto. Then compute the kernel of $f = \pi \circ i$.

3. a. Recall that if $p \in \mathbb{Z}$ is prime and $p|ab$ then either $p|a$ or $p|b$ (or both). Consider the ideal $I = p\mathbb{Z}$ ($p$ prime). Prove: if $ab \in I$ then either $a \in I$ or $b \in I$ (or both).

b. An ideal $I$ ($I \neq R$) in a commutative ring $R$ is called prime if for all $a, b \in R$ if $ab \in I$ then either $a \in I$ or $b \in I$. Prove: $I \subset R$ is a prime ideal if and only if $R/I$ has no zero divisors, i.e., is an integral domain.

c. Prove that the zero ideal $\{0\}$ in a ring $R$ is prime if and only if $R$ is an integral domain.

Hint: there are several ways to do this.

4. Suppose $S, T$ are two subrings of a ring $R$. Prove their intersection $S \cap T$ is also subring of $R$. 