

1 Prove that there is an addition and multiplication preserving bijection $\phi : \mathbb{C} \rightarrow \mathbb{R}[x]/(x^2 + 4)$.
Hint: mimic what we did on Monday with $\mathbb{R}[x]/(x^2 + 1)$. What do you need to change?

2 Suppose X is a set, \sim an equivalence relation and X/\sim is the set of the equivalence classes.
Consider the map/function $f : X \rightarrow X/\sim$, which takes $x \in X$ to its equivalence class. Prove that f is onto. Hint: it's very short.

3 Prove that $f(x) = 3x$ defines a bijection from \mathbb{R} to \mathbb{R} . Does it define a bijection from \mathbb{Z} to \mathbb{Z} ? If not, what goes wrong? If it does, what's the inverse?

4 Suppose A is not countable and $A \subset B$. Prove that B is not countable.

5 Prove that if $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, then so is their composition $f(g(x))$. Hint: Fix $x_0 \in \mathbb{R}$. Suppose $x_n \rightarrow x_0 \dots$

6 Prove that if p is prime then $\sqrt[3]{p}$ is irrational.
Hint: suppose $\sqrt[3]{p} = \frac{r}{s}$ where $r, s \in \mathbb{N}$ are relatively prime. Then $s^3 p = r^3$ and so $p|r^3 \dots$

7 Suppose $f : A \rightarrow B$ and $g : B \rightarrow A$ are two functions so that $f(g(b)) = b$ for all $b \in B$. Prove that f is onto and g is 1-1. Give an example where f is not 1-1 and g is not onto.

8 Which of the two statements below are true? Explain.

$$(\forall a \in \mathbb{R}) (\exists p(x) \in \mathbb{R}[x]) (p(a) = 0)$$

$$(\exists a \in \mathbb{R}) (\forall p(x) \in \mathbb{R}[x]) (p(a) = 0)$$

9 Prove that if $\{a_n\}$ is a sequence of real numbers with

$$|a_{n+1} - a_n| < \frac{1}{3^n}$$

for all $n \in \mathbb{N}$, then $\{a_n\}$ is Cauchy, hence converges.

10 Is the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^6 + y^4 = 1\}$$

the graph of a function from \mathbb{R} to \mathbb{R} ? Explain.

11 Find the greatest common divisor of 6188 and 4709 and write it as a linear combination of 6188 and 4709.

12 Prove that $10^{2007} - 1$ is divisible by 9. Hint: in $\mathbb{Z}/9$, $[10^{2007} - 1] = \dots$

13 Let c be a real number and $\{a_n\}$ a sequence converging to a number L . Prove, **using only the definition of the limit** of a sequence that $\{ca_n\}$ converges to cL . You **may not** quote theorems about limits in this problem.

14 Find and prove a formula for the sum :

$$1 + 5 + 9 + \dots + (4n + 1).$$

15 Prove that the limit of a sequence $\{a_n\}$ of real numbers is unique.

16 For what values of $z \in \mathbb{C}$ does the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converge?

17 Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ with the property that $\sum_{n=1}^{\infty} |a_n|$ does not converge.

18 Prove that $\gcd(x, n) = 1$ if and only if there is $y \in \mathbb{Z}$ so that $[x][y] = [1]$ in \mathbb{Z}/n .