Practice questions for the first midterm exam, Math 241

1. True/False questions, unless noted otherwise. No proofs required.

1. $|\vec{a} \cdot \vec{b}| \leq ||\vec{a}|| ||\vec{b}||$.

2. If $\vec{a} \cdot \vec{b} = 0$ then $\vec{a}$ and $\vec{b}$ are parallel to each other.

3. If $\vec{a} \times \vec{b} = \vec{0}$ then $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

4. The plane $2x - 3y + 5z = 6$ passes through the point $(2, 1, 1)$.

5. The plane $2x - 3y + 5z = 6$ and the line $(x, y, z) = (2t + 1, 2 - 3t, 5t)$ have a point in common.

6. If $\vec{a} \times \vec{b} = \vec{0}$ then $|\vec{a} \cdot \vec{b}| = ||\vec{a}|| ||\vec{b}||$.

7. All the level curves of the function $f(x, y) = x^2 - y^2$ are straight lines.

8. If for nonzero vectors $\vec{a}$ and $\vec{b}$ we have $||\vec{a} \times \vec{b}|| = \vec{a} \cdot \vec{b}$ then the angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$.

9. If $\vec{a}$ and $\vec{b}$ are parallel then $\vec{a} = \lambda \vec{b}$ for some scalar $\lambda$.

10. The map $f(x, y) = (x + y - 2, x - y)$ is linear.

11. If $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then $F$ is differentiable at every point and $DF(\vec{a})\vec{h} = F(\vec{h})$ for all $\vec{a}, \vec{h} \in \mathbb{R}^n$.

12. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function. If $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})$ exists and is finite, then $f$ is continuous at $\vec{a}$.

13. If the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $z = f(x, y)$ exists, then $f$ is differentiable.

14. If the function $F = (F_1, \ldots, F_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable, then the partials $\frac{\partial F_i}{\partial x_j}$ exist for all $1 \leq i \leq m, 1 \leq j \leq n$.

15. Suppose $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ is 0 if $(x, y) \rightarrow (0, 0)$ along the line $y = x$ and it is 1 if $(x, y) \rightarrow (0, 0)$ along the line $y = -x$. Then $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$
   
   (a) exists and equals 0
   (b) exists and equals 1/2
   (c) exists and equals 1
   (d) exists but its value cannot be determined from the information given
   (e) does not exist

16. If $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = \vec{v}$ and $\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) = \vec{w}$, then

   \[ \lim_{\vec{x} \rightarrow \vec{a}} (f(\vec{x}) + g(\vec{x})) = \vec{v} + \vec{w} \]

17. If $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear, $L(\vec{i}) = (1, 2, 3)$ and $L(\vec{j}) = (1, 0, 1)$ then $L(1, -1) =$?

2. Proofs.

1. Prove that for any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$,

   \[ ||\vec{v} - \vec{w}||^2 + ||\vec{v} + \vec{w}||^2 = 2(||\vec{v}||^2 + ||\vec{w}||^2) \]

   Hint: $||\vec{v} - \vec{w}||^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \ldots$
2. Is the map \( f(x, y) = (x + 2, xy) \) linear? Prove your answer.

3. Prove that if \( L, T : \mathbb{R}^n \to \mathbb{R}^m \) are two linear maps, then their sum \( L + T \) is also linear. The sum \( L + T \) is defined by \((L + T)(\vec{v}) = L(\vec{v}) + T(\vec{v})\).

4. Prove that the zero map \( O : \mathbb{R}^n \to \mathbb{R}^m \) that sends every vector to \( \vec{0} \) is linear. What do you need to check?

5. Prove using the definition that the function \( f(\vec{v}) = ||\vec{v}|| \) is continuous at \( \vec{0} \). That is, given \( \epsilon > 0 \) find \( \delta > 0 \) so that if \( ||\vec{v} - \vec{0}|| < \delta \) then \( ||f(\vec{v}) - f(\vec{0})|| < \epsilon \).

6. Suppose \( z = f(x, y) \) is a continuous function with \( f(0, 0) = 0 \). Prove that \( g(x, y) = ||(x, y)||f(x, y) \) is differentiable at zero with \( Dg(0, 0) = O \), the zero linear map.

Computational problems.

1. Write down an equation of the plane through the point \((1, 2, -3)\) normal to the line through the points \((0, 1, -1)\) and \((1, 0, 1)\).

2. Suppose \( z = f(x, y) \) is a differentiable function with \( f(1, 1) = 0 \) and \( Df(1, 1) = (1, -2) \). What is \( f(1.02, 0.99) \), approximately?

3. What is the derivative of the function \( f(x, y) = (xy, x^y) \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) at \((1, 1)\)?

4. Compute the determinant of the matrix \[
\begin{pmatrix}
1 & 2 & 0 \\
3 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix}
\]

5. Compute the cross product \((1, 3, 3) \times (2, 0, -1)\).