Practice Final

Note: This practice final has a bias towards things not covered on the midterms, since you have fewer materials from that section of the course to help you study. However, on the final all topics should be represented equally.

1. Find integers \( x \) and \( y \) such that
   \[
   13x + 15y = 1
   \]

2. (a) Let \( i \geq 1 \). Show that
   \[
   \frac{10^i - 1}{9}
   \]
   is an integer.
   (b) Let \( p \not\equiv 2,3,5 \) be a prime number. Show that \( p \) divides
   \[
   \frac{10^{p-1} - 1}{9}
   \]
   (c) Show that every prime \( p \not\equiv 2,3,5 \) divides a number of the form
   \[
   11 \cdots 1
   \]

3. Let \( p \) be an odd prime. Find a formula for the Legendre symbol
   \[
   \left( \frac{-2}{p} \right)
   \]

4. Let \( p \equiv 1 \mod 4 \) be a prime and \( a \) a quadratic residue mod \( p \). Decide with justification if then automatically \( p - a \) is quadratic residue mod \( p \).

5. (a) Calculate \( \phi(7!) \).
   (b) Suppose \( p \) and \( q \) are twin primes, i.e. \( q = p + 2 \). Show that
   \[
   \phi(q) = \phi(p) + 2
   \]
   (c) Suppose
   \[
   n = 2^x
   \]
   Find \( \nu(n) \) and \( \sigma(n) \).
6. Find the least non-negative residue of $3^{2011}$ modulo 22.

7. Suppose $p$ is prime and $n \geq 2$ and $a^{p^2} \equiv 1 \mod n$. Show that $\text{ord}_n a = p^2$ if and only if $a^p \not\equiv 1 \mod n$.

8. Let $n \geq 1$. Find $\gcd(n, 2n^2 + 1)$

9. Find the collection of all integers that are of the form $\text{ord}_{151}(a)$ where $a$ ranges through the integers co-prime to 151.

10. (a) State the Mobius inversion formula.
    (b) Show that for all $n \geq 1$ one has
        $$\sum_{d|n} \nu(d)\mu(n/d) = 1$$

11. (a) Find all primitive roots modulo 13.
    (b) How many primitive roots are there modulo 171?
    (c) How many primitive roots are there modulo 173?

12. How many primitive roots are there modulo $12^{100}$?

13. Find the order of 12 modulo 25.

14. Write down the continued fraction expansion for $\sqrt{29}$. Find its first five convergents.

15. Which quadratic number does the continued fraction $[4, \overline{2,1}]$ correspond to?

16. Find two continued fraction expansions for $\frac{13}{5}$. Are there others? Why or why not?

17. Show that $\frac{5042}{2911}$ is a convergent of $\sqrt{3} = 1.7320508075\ldots$.

18. Which of the following can be written as a sum of two squares? A sum of three squares? Four squares?
    (a) 39470
    (b) 55555
    (c) 34578
    (d) 12!
    (e) A number of the form $p^2 + 2$, where $p$ is a prime.

19. Suppose $x \in \mathbb{Z}_{>0}$ can be written as a sum of two squares. What is the necessary and sufficient condition on $y \in \mathbb{Z}_{>0}$ for $xy$ to be expressible as a sum of two squares?

20. Show that the area of any right triangle with all integer sides is divisible by 6.