Some Math 347 Solutions to Practice Problems for Midterm 1

1) True or False (justify for yourself, though you won’t have to justify T/F on the midterm):
   a) The integers \( \mathbb{Z} \) together with the usual addition and multiplication make up a field. F: 2 has no multiplicative inverse in \( \mathbb{Z} \).
   b) The set \( \{1\} \) has only one subset. F: it has two: itself and the empty set.
   c) If \( A \) and \( B \) are finite sets then the number of elements in \( A \cup B \) is \( |A| + |B| - |A \cap B| \). T: subtracting \( |A \cap B| \) accounts for those elements that are counted twice because they are in both sets.
   d) The converse of a statement is equivalent to the statement itself. F: consider the statement “if there is rain then there are clouds.”
   e) The contrapositive of a statement is equivalent to the statement itself. T
   f) \( a \) is a zero of a polynomial \( f(x) \) if and only if \( f(x) = (x-a)h(x) \) for some polynomial \( h(x) \). T (proven in class).

2) a) Prove that \( 5\sqrt{2} \) is irrational. You are allowed to assume that \( \sqrt{2} \) is irrational.

   Suppose \( 5\sqrt{2} \in \mathbb{Q} \). Also, we know that \( 1/5 \in \mathbb{Q} \). Since \( \mathbb{Q} \) is a field, it is closed under multiplication: in other words \( (5\sqrt{2}) \cdot (1/5) = \sqrt{2} \in \mathbb{Q} \). But we know that \( \sqrt{2} \notin \mathbb{Q} \), so we get a contradiction. Hence \( 5\sqrt{2} \notin \mathbb{Q} \).

   b) Prove that the sum of a rational number and an irrational number is irrational.

   Suppose \( R \) is a rational number and \( I \) is irrational. Suppose for contradiction that \( R + I = B \in \mathbb{Q} \). Then we have that \( I = B - R \), the sum of two rationals, which is rational since \( \mathbb{Q} \) is a field. But \( I \) is irrational and so we have a contradiction. Thus the sum of any rational and irrational number is irrational.

   c) Is the sum of two irrational numbers always irrational? Justify your answer.

   No. Counterexample: the rational number \( 0 = \sqrt{2} + (-\sqrt{2}) \) is a sum of two irrationals.

3) Let \( F = \{0, 1, 2, 3\} \) and let addition and multiplication be defined in such a way that \( 2 \cdot 0 = 0, 2 \cdot 1 = 2, 2 \cdot 2 = 0, \) and \( 2 \cdot 3 = 2 \). Can you define addition and multiplication for all of the other possible pairs of elements in \( F \) in such a way that what you get is a field? Justify your answer.

   No. The element 2 already has no multiplicative inverse since there is no \( a \in F \) such that \( 2 \cdot a = 1 \).

4) a) Let \( A = \{x \in \mathbb{Z} \mid x = 6u \text{ for some } u \in \mathbb{Z}\}, B = \{x \in \mathbb{Z} \mid x = 3u \text{ for some } u \in \mathbb{Z}\}, \) and \( C = \{x \in \mathbb{Z} \mid x = 2u \text{ for some } u \in \mathbb{Z}\} \). Show that \( A = B \cap C \).

   We first show that \( A \subset B \cap C \). To do this, consider \( a \in A \), meaning \( a = 6u \) for some \( u \in \mathbb{Z} \). Then \( a = 3 \cdot (2u) \) where \( 2u \in \mathbb{Z} \) and so we have \( a \in B \). At the same time, \( a = 2 \cdot (3u) \) where \( 3u \in \mathbb{Z} \) and so \( a \in C \). Hence
\[ a \in B \cap C. \]

Going the other way, we wish to show that \( B \cap C \subseteq A \). Suppose \( x \in B \cap C \). Then \( x = 2y = 3z \) for some integers \( y, z \). Since \( 2y \) is divisible by 2, we have \( 3z \) is divisible by 2, and hence \( z \) must be divisible by 2 as well (if you would like you can prove this statement). Hence \( z = 2u \) and so \( x = 3 \cdot 2u = 6u \) for some integer \( u \) and so \( x \in A \).

Having proven both containments we have that \( A = B \cap C \) must hold.

b) Find the power set of \( \{1, 2, 3, 4\} - \{\{1\}, 2, 16, \text{purple fish}\} \).

First of all, this set is simply \( \{1, 3, 4\} \) (note that the element \( \{1\} \) in the second set is different from the element 1 in the first set). The power set of this is the set of all of the following sets: the empty set, \( \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\} \).

5) Write down the negations of the following statements:

a) \( f(x, y) \neq 0 \) whenever \( x \neq 0 \) and \( y \neq 0 \).

There are some nonzero \( x \) and \( y \) such that \( f(x, y) = 0 \).

b) For all \( M \in \mathbb{R} \) there exists an \( x \in \mathbb{R} \) such that \( |f(x)| \geq M \).

There is some \( M \in \mathbb{R} \) for which \( |f(x)| < M \) for all \( x \in \mathbb{R} \).

c) For all \( M \in \mathbb{R} \) there exists an \( x \in \mathbb{R} \) such that for all \( y > x \) we have \( f(y) > M \).

There is an \( M \in \mathbb{R} \) such that for all \( x \in \mathbb{R} \) we have \( f(y) \leq M \) for some \( y > x \).

d) For all \( x \in \mathbb{R} \) there exists a \( y \in \mathbb{R} \) such that \( f(y) > f(x) \).

There is an \( x \in \mathbb{R} \) such that \( f(y) \leq f(x) \) for all \( y \in \mathbb{R} \).

e) For every \( \varepsilon > 0 \) there exists an \( x_0 \in \mathbb{R} \) such that \( |f(x)| < \varepsilon \) for all \( x > x_0 \).

There exists \( \varepsilon > 0 \) such that for any \( x_0 \in \mathbb{R} \) we have \( |f(x)| \geq \varepsilon \) for some \( x > x_0 \).

f) For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x) - f(x_0)| < \varepsilon \) whenever \( |x - x_0| < \delta \).

There exists some \( \varepsilon > 0 \) such that for any \( \delta > 0 \) we have that there is some \( x_0 \) with \( |x - x_0| < \delta \) but \( |f(x) - f(x_0)| \geq \varepsilon \).

6) a) Write down the following sum in summation notation: \( 1 + 4 + 9 + \cdots + n^2 \).

\[ \sum_{i=1}^{n} i^2 \]

b) Prove that the above sums to \( \frac{n(n+1)(2n+1)}{6} \).
We prove this by induction.

Base case:

\[ \sum_{i=1}^{1} i^2 = 1 = \frac{(1+1)(2+1)}{6} \]

Inductive hypothesis: suppose that for some \( n \geq 1 \) we have

\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

Inductive step: Consider

\[ \sum_{i=1}^{n+1} i^2 = n + 1 + \sum_{i=1}^{n} i^2 = (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \]

This is simply

\[ \frac{6(n+1)(n+1) + n(n+1)(2n+1)}{6} = \frac{(6n+6+2n^2+n)(n+1)}{6} = \frac{(n+1)(n+2)(2(n+1)+1)}{6} \]

as desired.

Hence, by induction, the statement is true.

7) Prove the following:

a) If \( x \) and \( y \) are two integers whose product is odd, then both must be odd.

Suppose for contradiction that \( xy \) is odd but one of them (without loss of generality, we can say \( x \)) is even. Then \( x = 2k \) for some integer \( k \) and so \( xy = 2ky \) where \( ky \) is an integer, so \( xy \) is even, a contradiction. Hence both \( x \) and \( y \) are odd.

b) If \( a \) and \( b \) are real numbers such that the product \( ab \) is an irrational number, then either \( a \) or \( b \) must be an irrational number.

We prove the contrapositive of this, which is equivalent: namely, we show that if both \( a \) and \( b \) are rational, then \( ab \) is rational. This simply follows from the fact that \( \mathbb{Q} \) is a field.

8) Show that there are no integers \( x \) and \( y \) such that \( x^2 - y^2 = 10 \).

Suppose there such integers \( x \) and \( y \). Then we have that either \( x, y \) are both odd or both even (since \( x^2 - y^2 = 10 \), which is even, but \( x^2 - y^2 \) is odd unless they are both odd or both even). Also, we have \( 10 = (x-y)(x+y) \), and note that \( x+y \) and \( x-y \) are both even in our case (odd minus odd and even minus even are both even). The product of two even numbers is divisible by 4 (shown in class), and so 10 is
divisible by 4, which it is not. Thus we have a contradiction and there are in fact no integers $x$ and $y$ such that $x^2 - y^2 = 10$. 