Some Math 347 Practice Problems for Midterm 1

Here are some problems to work on in preparation for the first midterm. It is not meant to represent what a midterm might look like, mainly because it is too long and some of the problems may be too involved for a midterm. However, doing them will help prepare you for the midterm. Note that it might be more useful for you to start working on these problems before you are completely prepared, since they will likely help you realize what you do and don’t understand. Solutions will be posted during the weekend before the test.

1) True or False (justify for yourself, though you won’t have to justify T/F on the midterm):
   a) The integers \( \mathbb{Z} \) together with the usual addition and multiplication make up a field.

   b) The set \( \{1\} \) has only one subset.

   c) If \( A \) and \( B \) are finite sets then the number of elements in \( A \cup B \) is \( |A| + |B| - |A \cap B| \).

   d) The converse of a statement is equivalent to the statement itself.

   e) The contrapositive of a statement is equivalent to the statement itself.

   f) \( a \) is a zero of a polynomial \( f(x) \) if and only if \( f(x) = (x - a)h(x) \) for some polynomial \( h(x) \).

2) a) Prove that \( 5\sqrt{2} \) is irrational. You are allowed to assume that \( \sqrt{2} \) is irrational.

   b) Prove that the sum of a rational number and an irrational number is irrational.

   c) Is the sum of two irrational numbers always irrational? Justify your answer.

3) Let \( F = \{0, 1, 2, 3\} \) and let addition and multiplication be defined in such a way that \( 2 \cdot 0 = 0, 2 \cdot 1 = 2, 2 \cdot 2 = 0, \) and \( 2 \cdot 3 = 2 \). Can you define addition and multiplication for all of the other possible pairs of elements in \( F \) in such a way that what you get is a field? Justify your answer.

4) a) Let \( A = \{x \in \mathbb{Z} \mid x = 6u \text{ for some } u \in \mathbb{Z}\} \), \( B = \{x \in \mathbb{Z} \mid x = 3u \text{ for some } u \in \mathbb{Z}\} \), and \( C = \{x \in \mathbb{Z} \mid x = 2u \text{ for some } u \in \mathbb{Z}\} \). Show that \( A = B \cap C \).

   b) Find the power set of \( \{1, 2, 3, 4\} - \{\{1\}, 2, 16, \text{purple fish}\} \).

5) Write down the negations of the following statements:

   a) \( f(x,y) \neq 0 \) whenever \( x \neq 0 \) and \( y \neq 0 \).

   b) For all \( M \in \mathbb{R} \) there exists an \( x \in \mathbb{R} \) such that \( |f(x)| \geq M \).

   c) For all \( M \in \mathbb{R} \) there exists an \( x \in \mathbb{R} \) such that for all \( y > x \) we have \( f(y) > M \).

   d) For all \( x \in \mathbb{R} \) there exists a \( y \in \mathbb{R} \) such that \( f(y) > f(x) \).

   e) For every \( \varepsilon > 0 \) there exists an \( x_0 \in \mathbb{R} \) such that \( |f(x)| < \varepsilon \) for all \( x > x_0 \).
f) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

6) a) Write down the following sum in summation notation: $1 + 4 + 9 + \cdots + n^2$.

   b) Prove that the above sums to $\frac{n(n+1)(2n+1)}{6}$.

7) Prove the following:
   a) If $x$ and $y$ are two integers whose product is odd, then both must be odd.
   b) If $a$ and $b$ are real numbers such that the product $ab$ is an irrational number, then either $a$ or $b$ must be an irrational number.

   (Hint: use contrapositive)

8) Show that there are no integers $x$ and $y$ such that $x^2 - y^2 = 10$. 