Math 250A: Sketch of proof that $\mathrm{PSL}_2(\mathbb{R})$ is simple

In this note we show that the projective special linear group $\mathrm{PSL}_2(\mathbb{R})$ (i.e. $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$) is simple. The proof generalizes easily to $\mathrm{PSL}_n(F) := \mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F))$ where F is a field with more than 3 elements. Note that in general $Z(\mathrm{SL}_n(F))$ is the set of scalar matrices, or diagonal matrices with the same value in each diagonal entry in $\mathrm{SL}_n(F)$.

Note that showing this is equivalent to showing that if $N \lhd \mathrm{SL}_2(\mathbb{R})$ then N is either trivial or $N = \{\pm I\}$, given the correspondence between subgroups of a group G and G/H where $H \trianglelefteq G$. With this in mind, consider the action of $G = \mathrm{SL}_2(\mathbb{R})$ on one-dimensional subspaces of $\mathbb{R}^2$ induced by left multiplication, and let S be the stabilizer of the subspace $\{(x,0) \mid x \in \mathbb{R}\}$ under this action. Note first of all that this action of G is doubly transitive: i.e. given one-dimensional subspaces V_1, V_2, W_1, W_2 where $V_1 \neq V_2$ and $W_1 \neq W_2$, there is an element of G which sends V_1 to W_1 and V_2 to W_2. To see this, note that V_i is spanned by some vector $\mathbf{v}_i \in \mathbb{R}^2$ and W_i is spanned by some vector $\mathbf{w}_i \in \mathbb{R}^2$ such that both $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ are bases for $\mathbb{R}^2$. We view the vectors $\mathbf{v}_i$ and $\mathbf{w}_i$ as column vectors in this note. Define two matrices $A = [\mathbf{v}_1\ \mathbf{v}_2]$ and $B = [\mathbf{w}_1\ \mathbf{w}_2]$ which both lie in $\mathrm{GL}_2(\mathbb{R})$. Let $d = \det(A)/\det(B)$ and let $D = d \cdot I$ where I is the identity matrix. Then clearly $BDA^{-1} \in \mathrm{SL}_2(\mathbb{R})$ and $BDA^{-1}\mathbf{v}_i = d\mathbf{w}_i$ so it sends V_1 to W_1 and V_2 to W_2 as desired.

We now show that if X is a doubly transitive G-set, then the stabilizer G_x of $x \in X$ is a maximal subgroup of G (i.e. there is no proper subgroup of G which properly contains G_x). First, note that G acts on the set of cosets G/G_x by left multiplication, and that X and G/G_x are isomorphic as G-sets (consider the map $\phi : G/G_x \to X$ where $\phi(gG_x) := gx$ and note that ϕ is a well-defined bijective map which is a G-set homomorphism). Suppose that G_x is not maximal and let $K < G$ be a proper subgroup of G properly containing G_x. Let $g, k \in G$ such that $g \notin K$ and $k \notin G_x$. Since X and G/G_x are isomorphic as G-sets, we have that G acts doubly transitively on G/G_x, and so there is an element $h \in G$ such that $hG_x = G_x$ and $h(kG_x) = gG_x$. This would mean that $h \in G_x$ and so $hk \in K$. At the same time, we get $g^{-1}hk \in G_x < K$ and so we get that $g \in K$, contrary to what we assumed. Thus G_x is a maximal subgroup of G.

To sum up, we have so far shown that the stabilizer S of the subspace $\{(x,0) \mid x \in \mathbb{R}\}$ under the above action of $\mathrm{SL}_2(\mathbb{R})$ on one-dimensional subspaces of $\mathbb{R}^2$ is a maximal subgroup of $\mathrm{SL}_2(\mathbb{R})$. Now, let

$$K = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$

and note that K is an abelian normal subgroup of S. We now show that if $N \lhd G$ then $N \leq S$.

Suppose $N \not\leq S$. Then we have that $S < SN \leq G$, and since S is maximal in G we must have $SN = G$. Let $\pi : G \to G/N$ be the canonical map, and note that $\pi(S) = SN/N = G/N = \pi(G)$, and $\pi(K) = KN/N = \pi(KN)$ where K is as above, so we have that $\pi(KN) \trianglelefteq \pi(G)$ since $K \trianglelefteq S$. Thus any conjugate of K is a subgroup of KN. But these conjugates can be easily seen to include both the subgroup K and the subgroup

$$K' = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$

which together generate G (you can work out the proof of this!). So we have that $KN = G$ and by the first isomorphism theorem we get that $G/N = KN/N \cong K/K \cap N$. Since K is abelian we have that $K/K \cap N$ and hence G/N is abelian, and so N must contain the commutator subgroup of G. Note that K is contained in the commutator of G. For example

$$\left[\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & k(a^2-1) \\ 0 & 1 \end{pmatrix}$$

which gives all elements of K as a and k vary. Similarly, K' is contained in the commutator of G and since K and K' generate G[1] we get that N contains G and so $N = G$. But we assumed that N is a proper normal subgroup of G and so we have a contradiction, so $N \leq S$.

So we have shown that N stabilizes the subspace $\{(x,0) \mid x \in \mathbb{R}\}$, and so $N = gNg^{-1}$ stabilizes $g\{(x,0) \mid x \in \mathbb{R}\}$ for all $g \in G$. Thus in particular it stabilizes the subspace $\{(0,y) \mid x \in \mathbb{R}\}$. The only matrices in G that stabilize both of these subspaces are $\pm I$ and so $N \leq \{\pm I\}$ as desired.

The proof that $\mathrm{PSL}_n(F)$ where F is a field with more than 3 elements is simple is basically identical, with a few more things to keep track of. For example, the group K should be replaced by the group of matrices with 1 on the diagonal and 0's everywhere else except for in the first row, while the role that the groups K and K' play in the proof that $N \leq S$ is played by the "root subgroups" X_{ij}: here X_{ij} is the group where all matrices have 1's on the diagonal and 0's everywhere else except for perhaps the ij-th entry. Also, our quest to show that any normal subgroup must be contained in $\{\pm I\}$ is replaced with the quest to show that any normal subgroup must consist only of scalar matrices.

[1]Note that we just showed that the commutator of $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SL}_2(\mathbb{R})$.