Math 250A: some practice problems for the final

1. (a) Show that there is a nonabelian subgroup T of $S_3 \times \mathbb{Z}_4$ of order 12 generated by elements a, b such that $|a| = 6$, $a^3 = b^2$, and $ba = a^{-1}b$.

   (b) Show that any group of order 12 with generators a, b such that $|a| = 6$, $a^3 = b^2$, and $ba = a^{-1}b$ is isomorphic to T.

2. Classify up to isomorphism all groups of order 18.

3. Show that if $N \trianglelefteq G$ and $N \cap [G, G] = 0$ then $N < Z(G)$.

4. Show that if H, K are solvable subgroups of G such that $H \trianglelefteq G$, then HK is a solvable subgroup of G.

5. Prove the fundamental theorem of arithmetic (unique prime factorization) using the Jordan-Hölder theorem applied to $\mathbb{Z}_n$.

6. Let R be a UFD and let $d \in R$ be nonzero. Show that there are only a finite number of distinct principal ideals in R which contain (d).

7. Show that if R is a commutative ring with unity such that every submodule of every free R-module is free, then R is a PID.

8. Let D be a PID and let A,B be cyclic D-modules of nonzero orders r and s, respectively, where r and s are not relatively prime. Show that the invariant factors of $A \oplus B$ are the least common multiple and greatest common divisor of r and s.

9. Determine the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$.

10. Show that no finite field K is algebraically closed.

11. Let F_1 be an algebraically closed field extension of a field K_1 and let F_2 be an algebraically closed field extension of a field K_2. Show that if $\mathrm{trdeg}(F_1/K_1) = \mathrm{trdeg}(F_2/K_2)$, then every isomorphism of fields $K_1 \cong K_2$ extends to an isomorphism $F_1 \cong F_2$.