Math 250A Homework 9, due 11/2/2012

1. Let K/F be a Galois extension. Two intermediate fields L_1, L_2 are called conjugate if there is $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(L_1) = L_2$. Characterize conjugate intermediate fields in terms of the corresponding subgroups of $\mathrm{Gal}(K/F)$.

2. Consider the quaternion group Q_8 which as a set is given by $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication determined by $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$. Let F be a field. Find a degree 4 polynomial $f(x) \in F[x]$ whose splitting field over F is Galois with Galois group isomorphic to Q_8 or show that no such polynomial exists.

[The following two exercises reprove one of the results we showed in class from a more high-brow point of view]

Let G be a group and M an abelian group. We call M a G-module if there is a map $G \times M \to M$ such that $1_g m = m$, $\sigma(\tau(m)) = (\sigma\tau)m$, $\sigma(m_1 + m_2) = \sigma(m_1) + \sigma(m_2)$ for all $m, m_1, m_2 \in M$ and $\sigma, \tau \in G$. Define the set of 1-cocyles of G with coefficients in M by

$$Z^1(G, M) := \{f : G \to M \big| f(\sigma\tau) = f(\sigma) + \sigma f(\tau) \text{ for all } \sigma, \tau \in G\}$$

The 1-coboundaries of G with coefficients in M are defined to be the

$$B^1(G, M) := \{f : G \to M \big| f(\sigma) = \sigma(a) - a \text{ for some } a \in M\}$$

3. Show that $Z^1(G, M)$ is an abelian group where $f + g$ is defined via point wise addition and show that $B^1(G, M)$ is a subgroup.

The first cohomology group of G with coefficients in M is defined to be

$$\mathrm{H}^1(G, M) := Z^1(G, M)/B^1(G, M)$$

4. Let K/F be a Galois extension.

a) Show that $K^\times$ is a $\mathrm{Gal}(K/F)$-module via the natural Galois action.

b) Show that $\mathrm{H}^1(\mathrm{Gal}(K/F), K^\times) = 0$.

c) Use the previous result to reprove the following theorem from class: If K/F is a cyclic Galois extension of degree n with $\mathrm{Gal}(K/F) = \langle\sigma\rangle$ and F contains a primitive n'th root of unity ζ_n then there exists $\alpha \in K^\times$ such that $\sigma(\alpha) = \zeta_n \alpha$.

5. Show that $\cos(\pi/9)$ is algebraic over $\mathbb{Q}$ and find $[\mathbb{Q}(\cos(\pi/9) : \mathbb{Q}]$.

6. Show that $\mathbb{Q}(\cos(2\pi/n))$ is Galois over $\mathbb{Q}$ for every $n \in \mathbb{Z}^{\geq 1}$. Decide if the same holds for $\sin(2\pi/n)$.

7. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$.

a) Show that the Frobenius map $\mathrm{Fr}(x) = x^p$ is in $\mathrm{Aut}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and show that Fr is of infinite order.

b) Find $\sigma \in \mathrm{Aut}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ such that $\sigma \notin \langle \mathrm{Fr} \rangle$.

8. Let K/F be a Galois extension and L an intermediate field. Let $N \subseteq K$ be the normal closure of L over F. Prove that

$$\mathrm{Gal}(K/N) = \bigcap_{\sigma \in \mathrm{Gal}(K/F)} \sigma \mathrm{Gal}(K/L) \sigma^{-1}$$

9. Let K/F be a Galois extension and L an intermediate field and let $H = \mathrm{Gal}(K/L)$. Consider the normalizer $N_{\mathrm{Gal}(K/F)}(H)$ and let L_0 denote its fixed field.

a) Prove that L/L_0 is a Galois extension.

b) Prove that if M is an intermediate field of L/F such that L/M is a Galois extension then $M \supseteq L_0$.

10. Throughout let G be a finite abelian group. Let $\exp(G)$ (the "exponent") denote the least common multiple of the orders of the elements of G.

a) Show that there exists an element of G of order $\exp(G)$ and hence that $\exp(G)$ is the maximum of all the orders of elements in G.

b) Deduce that G is cyclic if and only if

$$\exp(G) = \#G$$

c) Let F be a field and H a finite subgroup of the group $F^{\times}$. Show that H is cyclic.

d) Let F be a field and suppose that $\mathrm{char}(F) \nmid n$. Show that there exists a primitive n'th root of unity in some finite extension of F.