1. Let $K/F$ be a Galois extension. Two intermediate fields $L_1, L_2$ are called conjugate if there is $\sigma \in \text{Gal}(K/F)$ such that $\sigma(L_1) = L_2$. Characterize conjugate intermediate fields in terms of the corresponding subgroups of $\text{Gal}(K/F)$.

2. Consider the quaternion group $Q_8$ which as a set is given by $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication determined by $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$. Let $F$ be a field. Find a degree 4 polynomial $f(x) \in F[x]$ whose splitting field over $F$ is Galois with Galois group isomorphic to $Q_8$ or show that no such polynomial exists.

[The following two exercises reprove one of the results we showed in class from a more high-brow point of view]

Let $G$ be a group and $M$ an abelian group. We call $M$ a $G$-module if there is a map $G \times M \to M$ such that $1_g m = m$, $\sigma(\tau(m)) = (\sigma\tau)m$, $\sigma(m_1 + m_2) = \sigma(m_1) + \sigma(m_2)$ for all $m, m_1, m_2 \in M$ and $\sigma, \tau \in G$. Define the set of 1-cocyles of $G$ with coefficients in $M$ by

$$Z^1(G, M) := \{ f : G \to M | f(\sigma\tau) = f(\sigma) + \sigma f(\tau) \text{ for all } \sigma, \tau \in G \}$$

The 1-coboundaries of $G$ with coefficients in $M$ are defined to be the

$$B^1(G, M) := \{ f : G \to M | f(\sigma) = \sigma(a) - a \text{ for some } a \in M \}$$

3. Show that $Z^1(G, M)$ is an abelian group where $f + g$ is defined via point wise addition and show that $B^1(G, M)$ is a subgroup.

The first cohomology group of $G$ with coefficients in $M$ is defined to be

$$H^1(G, M) := Z^1(G, M)/B^1(G, M)$$

4. Let $K/F$ be a Galois extension.

a) Show that $K^\times$ is a $\text{Gal}(K/F)$-module via the natural Galois action.

b) Show that $H^1(\text{Gal}(K/F), K^\times) = 0$.

c) Use the previous result to reprove the following theorem from class: If $K/F$ is a cyclic Galois extension of degree $n$ with $\text{Gal}(K/F) = \langle \sigma \rangle$ and $F$ contains a primitive $n$’th root of unity $\zeta_n$ then there exists $\alpha \in K^\times$ such that $\sigma(\alpha) = \zeta_n \alpha$. 

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5. Show that \( \cos(\pi/9) \) is algebraic over \( \mathbb{Q} \) and find \([\mathbb{Q}(\cos(\pi/9) : \mathbb{Q})]\).

6. Show that \( \mathbb{Q}(\cos(2\pi/n)) \) is Galois over \( \mathbb{Q} \) for every \( n \in \mathbb{Z} \geq 1 \). Decide if the same holds for \( \sin(2\pi/n) \).

7. Let \( \overline{\mathbb{F}}_p \) be an algebraic closure of \( \mathbb{F}_p \).
   
   a) Show that the Frobenius map \( \text{Fr}(x) = x^p \) is in \( \text{Aut}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \) and show that \( \text{Fr} \) is of infinite order.
   
   b) Find \( \sigma \in \text{Aut}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \) such that \( \sigma \notin \langle \text{Fr} \rangle \).

8. Let \( K/F \) be a Galois extension and \( L \) an intermediate field. Let \( N \subseteq K \) be the normal closure of \( L \) over \( F \). Prove that

\[
\text{Gal}(K/N) = \bigcap_{\sigma \in \text{Gal}(K/F)} \sigma \text{Gal}(K/L) \sigma^{-1}
\]

9. Let \( K/F \) be a Galois extension and \( L \) an intermediate field and let \( H = \text{Gal}(K/L) \). Consider the normalizer \( N_{\text{Gal}(K/F)}(H) \) and let \( L_0 \) denote its fixed field.
   
   a) Prove that \( L/L_0 \) is a Galois extension.
   
   b) Prove that if \( M \) is an intermediate field of \( L/F \) such that \( L/M \) is a Galois extension then \( M \supseteq L_0 \).

10. Throughout let \( G \) be a finite abelian group. Let \( \text{exp}(G) \) (the “exponent”) denote the least common multiple of the orders of the elements of \( G \).
   
   a) Show that there exists an element of \( G \) of order \( \text{exp}(G) \) and hence that \( \text{exp}(G) \) is the maximum of all the orders of elements in \( G \).
   
   b) Deduce that \( G \) is cyclic if and only if

\[
\text{exp}(G) = \#G
\]

   c) Let \( F \) be a field and \( H \) a finite subgroup of the group \( F^\times \). Show that \( H \) is cyclic.
   
   d) Let \( F \) be a field and suppose that \( \text{char}(F) \nmid n \). Show that there exists a primitive \( n \)'th root of unity in some finite extension of \( F \).