Math 250A Homework 8, due 10/26/2012

1) Let $K/F$ be a Galois extension with $[K : F] = n$. Show that if $p | n$ is a prime then there is a subfield $L$ of $K$ with $[K : L] = p$.

2) Let $K/F$ be a Galois extension with $\text{Gal}(K/F) \cong A_4$. Show that there is no intermediate field $M$ of the extension $K/F$ such that $[M : F] = 2$.

3) Show that if $K/F$ is a Galois extension such that there are no proper intermediate fields between $K$ and $F$, then $[K : F]$ is a prime number. Is this still true if $K/F$ is not a Galois extension?

4) Let $K/F$ be a Galois extension and $\alpha \in K$ and $H = \text{Gal}(K/F(\alpha))$. Let $[K : F] = n$ and $[F(\alpha) : F] = r$. Suppose that $\{\tau_1, \cdots, \tau_r\}$ is a set of left coset representatives of $H$ in $\text{Gal}(K/F)$. Show that the minimal polynomial of $\alpha$ over $F$ is given by

$$m(x) = \prod_{i=1}^{r} (x - \tau_i(\alpha))$$

and show that

$$\prod_{\sigma \in \text{Gal}(K/F)} (x - \sigma(\alpha)) = m(x)^{n/r}$$

5) Let $K/F$ be a Galois extension with $\text{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\text{char}(F) \neq 2$. Show that $K = F(\sqrt{\alpha}, \sqrt{\beta})$ for some $\alpha, \beta \in F$.

6) Let $K$ be the splitting field of $x^8 - 1$ over $\mathbb{Q}$. Find $\text{Gal}(K/\mathbb{Q})$ and describe all intermediate fields of $K/\mathbb{Q}$.

7) Let $S = \{ \sqrt{p} \mid p \text{ a prime}\}$ and $K = \mathbb{Q}(S)$. For $\sigma \in \text{Aut}(K/\mathbb{Q})$ define

$$Y_\sigma = \{ \sqrt{p} \mid \sigma(\sqrt{p}) = -\sqrt{p}\}$$

Show the following:

(i) $K$ is normal and separable over $\mathbb{Q}$
(ii) If $Y_\sigma = Y_\tau$, then $\sigma = \tau$
(iii) If $Y$ is a subset of $S$ then there exists $\sigma \in \text{Aut}(K/\mathbb{Q})$ such that $Y = Y_\sigma$
(iv) Let $P(S)$ denote the set of all subsets of $S$. Show that

$$[K : \mathbb{Q}] = |S|$$

and

$$|\text{Aut}(K/\mathbb{Q})| = |P(S)|$$

[Note that it follows that $|\text{Aut}(K/\mathbb{Q})| > [K : \mathbb{Q}]$]

8) Let $K$ be a subfield of $\mathbb{C}$ such that $K/\mathbb{Q}$ is a Galois extension. Let $c \in \text{Aut}(\mathbb{C})$ be complex conjugation.

a) Show that $c(K) = K$ and the restriction $c|_K$ of $c$ to $K$ is an element of $\text{Gal}(K/\mathbb{Q})$.

b) Show that $\mathcal{F}(c|_K) = K \cap \mathbb{R}$ and $[K : K \cap \mathbb{R}] \leq 2$.

c) Give an example of $K$ where $[K : K \cap \mathbb{R}] = 1$ and an example of $K$ where $[K : K \cap \mathbb{R}] = 2$.

9) Let $k$ be a field of characteristic $p > 0$, let $K = k(x, y)$ be the rational function field in two variables over $k$, and let $F = k(x^p, y^p)$.


b) Prove that $K^p \subseteq F$ (see Homework 7 for definition of $K^p$).

c) Prove that there is no $\alpha \in K$ with $K = F(\alpha)$.

d) Exhibit an infinite number of intermediate fields of $K/F$.

10) Let $K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ and let $F = \mathbb{Q}$. Show directly that there exists a $\sigma \in \text{Aut}(K/F)$ such that $\sigma(\sqrt[3]{2}) = \zeta_3 \sqrt[3]{2}$ and $\sigma(\zeta_3) = \zeta_3^2$. 