Math 250A Homework 7, due 10/12/2012

1) From Lang, Ch. V: problems 10, 13.

2) Find the degree of the splitting field of $x^6 + 1$ over $\mathbb{Q}$. How about over $\mathbb{F}_2$?

3) a) Let $F$ be a field, let $f(x) \in F[x]$ be a polynomial of prime degree. Suppose for every field extension $K$ of $F$ that if $f$ has a root in $K$, then $f$ splits over $K$. Prove that either $f$ is irreducible over $F$ or $f$ has a root (and hence splits) in $F$.

   b) Give two examples of situations in which the hypotheses in (a) hold for characteristic $p > 0$ fields, and one example of a situation in which the hypotheses in (a) hold in a characteristic 0 field.

4) Let $F$ be a field. Show that the rational function field $F(x)$ is not algebraically closed.

5) Let $F$ be a finite extension of $\mathbb{Q}$. Show that $F$ is not algebraically closed.

6) Let $F$ be a field of characteristic $p$.
   a) Let $F^p = \{ a^p | a \in F \}$. Show that $F^p$ is a subfield of $F$.
   b) If $F = \mathbb{F}_p(x)$ is the rational function field in one variable over $\mathbb{F}_p$, determine $[F : F^p]$.

7) Show that every element of a finite field is a sum of two squares.

8) Let $f(x)$ be an irreducible polynomial over $F$ of degree $n$ and let $K$ be a field extension of $F$ with $[K : F] = m$. If $\gcd(n,m) = 1$, show that $f$ is irreducible over $K$.

9) Let $K$ and $L$ be extensions of $F$. Show that $KL$ is normal over $F$ if both $K$ and $L$ are normal over $F$. Is the converse true?