Math 250A Homework 6, due 10/5/2012

1) Let A be a ring and let F be a field. If A is also an F-vector space and $\alpha(rs) = (\alpha r)s = r(\alpha s)$ for all $\alpha \in F$ and all $r, s \in A$, then A is called an *F-algebra*. If A is an F-algebra, show that A contains an isomorphic copy of F. Also, show that if K is a field extension of F, then K is an F-algebra.

2) Let $F \subseteq L \subseteq K$ be fields. Show that

$$\operatorname{trdeg}(K/F) = \operatorname{trdeg}(K/L) + \operatorname{trdeg}(L/F).$$

3) We say a field K is *finitely generated* over a field F if $K = F(\alpha_1, \ldots, \alpha_k)$ for some $\alpha_i \in K$. Show that if $F \subseteq L \subseteq K$ are fields and K is finitely generated over F then L is finitely generated over F.

4) Let K be an algebraically closed field, and let F be a subfield of K. Let $\phi : K \to K$ be a ring homomorphism such that $\phi(\alpha) = \alpha$ for all $\alpha \in F$, and let $\operatorname{trdeg}(K/F)$ be finite. Show that ϕ is a bijection.

5) Let K be an algebraically closed field, and let F be a subfield of K with $\operatorname{trdeg}(K/F)$ infinite. Show that there is a ring homomorphism $\phi : K \to K$ such that $\phi(\alpha) = \alpha$ for all $\alpha \in F$ which is not surjective.

6) Let $K = \mathbb{R}(x)(\sqrt{-1-x^2})$. Show that $[K : \mathbb{R}(x)] = 2$ and that there is no $t \in K$ such that $K = \mathbb{R}(t)$.

7) Let x be transcendental over $\mathbb{C}$ and let K be the algebraic closure of $\mathbb{C}(x)$. Prove that $K \cong \mathbb{C}$.

8) Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$ and let $m(x)$ denote its minimal polynomial over $\mathbb{Q}$. Suppose $\beta \in \mathbb{C}$ is some other root of $m(x)$. Show that the map $\sigma : \mathbb{Q}(\alpha) \to \mathbb{C}$ which for every $f \in \mathbb{Q}[x]$ takes $f(\alpha)$ to $f(\beta)$ is a well defined field homomorphism such that $\sigma(x) = x$ for all $x \in \mathbb{Q}$.

9) Let F be a field of characteristic $p > 0$ and let $a \in F$ be an element which is not a p'th power of an element in F. Show that the polynomial $x^p - a \in F[x]$ is irreducible over F.

10) Let F be either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Decide in each case if F has more than one automorphism.

11) Let A be an integral domain with field of fractions F. Show that if $\sigma : A \to A$ is a ring automorphism then the map $\frac{a}{b} \mapsto \frac{\sigma(a)}{\sigma(b)}$ for $a, b \in A$ yields a ring automorphism of F.

12) Let $K = k(x_1, \ldots, x_n)$ be the field of rational functions in n variables over a field k. Show that the definition

$$\sigma\left(\frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}\right) = \frac{f(x_{\sigma(1)},\ldots,x_{\sigma(n)})}{g(x_{\sigma(1)},\ldots,x_{\sigma(n)})}$$

makes a permutation $\sigma \in S_n$ into a field automorphism of K.