Math 250A Homework 5, due 9/28/2012

1) Do problem 11 from Chapter IV in the book.

2) Show that the quotient of a PID by a prime ideal is again a PID.

3) Let D be an integral domain. Show that the following two conditions imply that D is a PID:

- Any two nonzero elements $a, b \in D$ have a g.c.d. which can be written in the form $ra + sb$ for some $r, s \in D$
- If $a_1, a_2, a_3, \dots$ are nonzero elements of D such that $a_{i+1} | a_i$ for all i, then there is a positive integer k such that $a_i = u_i a_k$, where $u_i \in D$ is a unit, for all $i \geq k$

4) Let D be an integral domain and suppose that every prime ideal is principal. In this exercise you will show that D must be a PID. Let S be the set of non-principal ideals in D. If S is nonempty, you may assume that S contains a maximal element.

a) Let S be as above, suppose S is nonempty, and let I be a maximal element in S. Let $a, b \in D$ such that $ab \in I$ but $a \notin I$, $b \notin I$. (why do such elements a, b exist?) Let $I_a = I + (a)$ and let $I_b = I + (b)$. Define $J = \{x \in D \mid xI_a \subseteq I\}$. Show that I_a and J are principal and that $I \subset I_b \subseteq J$. If $I_a = (\alpha)$ and $J = (\beta)$, show that $I_a J = (\alpha\beta) \subseteq I$.

b) Let I, I_a, J, α be as above. If $x \in I$ show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_a J$ is principal and conclude that D is a PID.

5) An integral domain D in which any ideal generated by two elements is principal is called a *Bezout domain*.

a) Prove that an integral domain D is a Bezout domain if and only if every pair of elements $a, b \in D$ has a g.c.d. in D which can be written as $ax + by$ for some $x, y \in D$.

b) Show that every finitely generated ideal of a Bezout domain is principal. *(however not every Bezout domain is a PID, which you need not show here)*

6) a) Show that if D is a UFD and $a, b \in D$ are relatively prime such that $ab = c^n$ for some $c \in D$ then there are units $u_1, u_2 \in D$ and elements $a', b' \in D$ such that $a = u_1 a'^n$ and $b = u_2 b'^n$.

b) Use (a) to show that the equation $y^2 + y = x^3$ has only two integral solutions.

c) *(Optional! Only for those of you who have time on their hands and like number theory a lot)* Show that the equation $y^2 = x^3 - 2$ has only two integral solutions. You can use the fact that $\mathbb{Z}[\sqrt{-2}]$ is a UFD.

7) a) Show that every Artinian integral domain is a field.

b) Deduce from (a) that every prime ideal in an Artinian ring is maximal.

8) Show that if R is Noetherian and $\phi : R \to R$ is a surjective homomorphism, then ϕ is also injective.