Math 250A Homework 4, due 9/21/2012

1) Do problems 22, 41 from chapter I and problems 1, 9, 10 from Chapter II in the book. You may use exercise I.40 to complete exercise I.41.

2) Show that if G is a finite group which has only one subgroup of order d for every $d \mid |G|$ then G is cyclic. *(Hint: use a Sylow theorem to reduce this problem to the case where G is a p-group first)*

3) a) Let $N \lhd G$ where G is a finite group and let p be a prime dividing $|N|$. Show that for any p-Sylow subgroup P of G we have that $P \cap N$ is a p-Sylow subgroup of N and that all p-Sylow subgroups of N arise in this way. Thus the number of p-Sylow subgroups of N is at most the number of p-Sylow subgroups of G.

b) Give a counterexample to the statement in (a) with $N \lhd G$ replaced by $N < G$.

4) a) Show that $\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5) \cong \mathrm{GL}_2(\mathbb{F}_5)$ and that $|\mathrm{GL}_2(\mathbb{F}_5)| = 480$.

b) Give an example of a nonabelian group of order 75.

c) For this problem, it might be helpful to know the following generalization of problem 6 from the last homework:

Let K be a cyclic group and let H be an arbitrary group. Let $\phi : K \to Aut(H)$ and $\psi : K \to Aut(H)$ be injective homomorphisms such that $\psi(K)$ is conjugate to $\phi(K)$ in $Aut(H)$. Then $H \rtimes_\phi K \cong H \rtimes_\psi K$.

With this in mind, classify all groups of order 75.

5) Show that a subring of a Noetherian ring need not be Noetherian.

6) Let R be a nonunital ring (a ring as defined in Lang, with no unity). Define the operation $*$ on R by

$$r * s = r + s + rs$$

and define an element $r \in R$ *quasi-invertible* if $r * s = 0 = s * r$ for some $s \in R$. Let $QU(R)$ denote the set of quasi-invertible elements in R.

a) Show that $QU(R)$ is a group under $*$.

b) If R is a ring (with unity) and $U(R)$ is the set of units in R, show that there is a group homomorphism $\phi : QU(R) \to U(R)$ given by $\phi(r) = 1 + r$.