1) Do problems 22, 41 from chapter I and problems 1, 9, 10 from Chapter II in the book. You may use exercise I.40 to complete exercise I.41.

2) Show that if $G$ is a finite group which has only one subgroup of order $d$ for every $d | |G|$ then $G$ is cyclic. (*Hint: use a Sylow theorem to reduce this problem to the case where $G$ is a $p$-group first*)

3) a) Let $N \triangleleft G$ where $G$ is a finite group and let $p$ be a prime dividing $|N|$. Show that for any $p$-Sylow subgroup $P$ of $G$ we have that $P \cap N$ is a $p$-Sylow subgroup of $N$ and that all $p$-Sylow subgroups of $N$ arise in this way. Thus the number of $p$-Sylow subgroups of $N$ is at most the number of $p$-Sylow subgroups of $G$.

b) Give a counterexample to the statement in (a) with $N \triangleleft G$ replaced by $N < G$.

4) a) Show that $\text{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5) \cong \text{GL}_2(\mathbb{F}_5)$ and that $|\text{GL}_2(\mathbb{F}_5)| = 480$.

b) Give an example of a nonabelian group of order 75.

c) For this problem, it might be helpful to know the following generalization of problem 6 from the last homework:

Let $K$ be a cyclic group and let $H$ be an arbitrary group. Let $\phi : K \to \text{Aut}(H)$ and $\psi : K \to \text{Aut}(H)$ be injective homomorphisms such that $\psi(K)$ is conjugate to $\phi(K)$ in $\text{Aut}(H)$. Then $H \rtimes_{\phi} K \cong H \rtimes_{\psi} K$.

With this in mind, classify all groups of order 75.

5) Show that a subring of a Noetherian ring need not be Noetherian.

6) Let $R$ be a nonunital ring (a ring as defined in Lang, with no unity). Define the operation $\ast$ on $R$ by

$$r \ast s = r + s + rs$$

and define an element $r \in R$ quasi-invertible if $r \ast s = 0 = s \ast r$ for some $s \in R$. Let $QU(R)$ denote the set of quasi-invertible elements in $R$.

a) Show that $QU(R)$ is a group under $\ast$. 
b) If $R$ is a ring (with unity) and $U(R)$ is the set of units in $R$, show that there is a group homomorphism $\phi : QU(R) \to U(R)$ given by $\phi(r) = 1 + r$. 