Math 250A Homework 3, due 9/14/2012

1) Do problems 19, 21, 25, 30 from chapter I in the book.

2) Let G be a finite group, and let r be the number of conjugacy classes of G. Show that

$$|\{(a,b) \in G \times G \mid ab = ba\}| = r|G|.$$

3) Let G be a finite group, let $N \trianglelefteq G$, and let P be a Sylow subgroup of N. Show that $G = N_G(P)N$, where $N_G(P)$ denotes the normalizer of P in G.

4) a) Let G be a group such that p, q are two distinct prime divisors of $|G|$. Suppose that P is the only p-Sylow subgroup of G and Q is the only q-Sylow subgroup of G. Show that the elements of P commute with the elements of Q.

b) Show that any group of order 45 is abelian.

5) Show that every Sylow subgroup is normal in G if and only if G is the direct product of its Sylow subgroups.

6) Let H be a cyclic group and let N be an arbitrary group. If ϕ and ψ are injective homomorphisms from H to $\mathrm{Aut}(N)$ such that $\phi(H) = \psi(H)$, show that $N \rtimes_\phi H \cong N \rtimes_\psi H$.