Math 250A Homework 12b, due 11/30/2012

1. Show that if a module M over a PID D has an annihilator $\nu \in D$ such that $\nu = \kappa\lambda$, where κ and λ are relatively prime, then M is the direct sum of its submodules $M^{(\kappa)}$ and $M^{(\lambda)}$, with the notation used in class.

2. Let D be a PID. For $p \in D$ prime, we define a *p-module* to be a D-module in which every element has order some power of p. We call a D-module *primary* if it is a p-module for some prime $p \in D$. Denote the set of all elements of order some power of the prime p in M by $T_p(M)$.

   (a) Show that $T_p(M)$ is the largest p-submodule of M.

   (b) Show that any finitely generated torsion D-module is a direct sum of primary modules. More explicitly, show that if such a module has minimal annihilator $\nu = p_1^{e_1} \cdots p_k^{e_k}$ where p_i are all primes in D which are not unit multiples of each other, then
   $$M \cong T_{p_1}(M) \oplus \cdots \oplus T_{p_k}(M).$$

3. Show that any finitely generated torsion module M over a PID D is isomorphic to a direct sum of primary cyclic modules. Show also that the list of orders of these primary cyclic modules is a list of powers of primes and that any other isomorphism of M to a direct sum of primary cyclic modules must have the same list of powers of primes associated to it, except perhaps for a permutation of the list or a replacement of each prime by a unit-multiple of the prime.

4. Let F be a field, and recall that to every $B \in M_{n\times n}(F)$ we can associate an $F[x]$-module structure of F^n by letting x act as B. Show that F^n regarded as an $F[x]$-module in this way is a finitely generated torsion module.

5. Let $p = (x - \lambda)$ for some $\lambda \in F$. Let C be a cyclic p-module of order p^e for some integer $e \geq 1$. We may think of C as $F[x]/((x-\lambda)^e)$. Show that C is an e-dimensional vector space over F. Furthermore, show that there is a basis of C over F such that C can be regarded as an $F[x]$-module obtained by letting x act as the matrix
$$\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & 0 \\ & & 1 & \lambda \end{pmatrix}$$

6. Explain how the above implies the theorem on the Jordan Canonical Form as presented in class.