Math 250A Homework 12a, due 11/26/2012, to be collected on 11/30/2012

1. Let M be an R-module, and suppose there is a surjective R-linear homomorphism $\phi : M \to R$. Show that
$$M \cong \ker(\phi) \oplus R.$$

2. Let M be an R module and let N be a submodule of M. Prove that if M/N and N are finitely generated then M is finitely generated.

3. Recall that a module M is called *irreducible* if the only submodules it has are M and 0.

   (a) Prove that if R is commutative, then M is irreducible iff $M \cong R/I$ as R-modules for some maximal ideal I of R.

   (b) If M and N are irreducible R-modules, prove that any nonzero R-module homomorphism from M to N is an isomorphism.

4. Prove that an R-module N satisfies the ACC for submodules iff there is in every nonempty set U of submodules of N a maximal element M (here maximal means M is not properly contained in any other element of U).

5. Let C be a cyclic D-module of order μ, where D is a PID.

   (a) Prove that every submodule of C is cyclic with order a divisor of μ.

   (b) For each principal ideal (λ) of D with $(\lambda) \supset (\mu)$, show that C has exactly one submodule which is cyclic of order λ.

6. If $D = F[x]$, with F a field, show that a cyclic D-module of order $f(x) \in F[x]$ is also a vector space over F of dimension the degree of the polynomial of f.