Math 250A Homework 11, due 11/16/2012

1. Let $c \in \mathbb{R}$ be a root of an irreducible quartic over $\mathbb{Q}$. Let N be the normal closure of $\mathbb{Q}(c)/\mathbb{Q}$.

   (a) If $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to either D_4 or a group of order 4, show that c is constructible.

   (b) If $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to either A_4 or S_4, show that c is not constructible.

2. In the definition of a module, show that the commutativity assumption for M is redundant (it is implied by the other properties).

3. For any natural number $n > 1$, show that an additive abelian group A in which $na = 0$ for all $a \in A$ can be regarded as a $\mathbb{Z}_n$-module.

4. (a) Let R be a commutative ring, let M be an R-module, and let α be an endomorphism of M. Show that M becomes an $R[x]$-module by the rule

   $$(a_0 + a_1 x + \cdots + a_k x^k)m = a_0 m + \alpha(a_0 m) + \cdots + \alpha^k(a_k m).$$

   Show also that if M is an $R[x]$-module then M is an R-module, and the map α given by $\alpha(m) = xm$ is an R-module endomorphism of M. Informally, x is said to *act as* α.

   (b) Suppose that M and N are $R[x]$-modules, with x acting as α and β, respectively. Prove that $\pi : M \to N$ is an $R[x]$-module homomorphism if and only if it is an R-module homomorphism with $\pi\alpha = \beta\pi$.

5. Let M be $\mathbb{C}^3$ made into a $\mathbb{C}[x]$-module using the matrix

   $$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   as seen in class (essentially, take the formulation in part (a) of the previous problem and let α be left multiplication by A). For any vector $v \in \mathbb{C}^3$ let $L(v)$ be the $\mathbb{C}[x]$-submodule generated by v. Write L_0 for the special submodule given by $(1, 0, 0)^t$. Show that for any $v \neq 0$, we have $L_0 \subseteq L(v)$. Find all v with $\dim(L(v)) = 2$.

6. We say an R-module is *irreducible* if it is nonzero and has no submodules other than 0 and itself. Show that the only irreducible $\mathbb{Z}$-modules are of the form $\mathbb{Z}/p\mathbb{Z}$ where p is prime.

7. A submodule M' of an R-module M is called *maximal* if $M' \neq M$ and there is no submodule L of M such that $M' \subset L \subset M$. Show that any proper submodule of a finitely generated R-module is contained in a maximal submodule.

8. In this exercise you are not to assume the fundamental theorem of finitely generated abelian groups. Let M be an abelian group and suppose $cM = 0$ for some positive integer c. Write the prime factorization of c in the form

$$c = q_1 \cdots q_k$$

where $p_1, \ldots, p_k$ are distinct primes and $q_i = p_i^{r(i)}$ for each i. Denote

$$\hat{q}_i = q_1 \cdots q_{i-1} q_{i+1} \cdots q_k$$

for all i. Show that $M = M_1 \oplus \cdots \oplus M_k$ with $M_i = \hat{q}_i M$ for each i.

9. In a 3-dimensional vector space V over $\mathbb{R}$, let D be the submodule of all vectors on a given line through the origin. Describe geometrically the quotient module V/D.

10. Let A, A' be R-modules, let $\operatorname{Hom}_R(A, A')$ denote the set of R-linear maps $A \to A'$. Show that $\operatorname{Hom}_R(A, A')$ is an R-module. Let $R = \mathbb{Z} \times \mathbb{Z}$. Give an example of abelian groups A, A' which are R-modules, such that $\operatorname{Hom}_R(A, A') \neq \operatorname{Hom}_{\mathbb{Z}}(A, A')$.