You might have noticed that most of the theorems we prove about groups have a very general ring to them: they often involve some arbitrary group $G$, with no particular special properties. This is one of the great things about groups: even though many of them seem to be completely different from one another, they all possess certain nice properties which might not be immediately evident from studying a specific group alone. Sometimes, groups appear to be just sets with some binary operation but in fact most groups are in some sense much more than that. One basic class of groups in which this “much more” is readily seen is in groups of symmetry of certain objects (symmetries of tessellations, of polyhedra, etc). In this note we’ll study one such symmetry group: the dihedral group $D_n$.

Suppose you draw a regular $n$-gon (meaning a convex polygon with $n$ sides of equal length and $n$ angles of the same size) on the floor, and cut out a polygon of exactly the same size and shape from paper. You also label each vertex of your paper polygon (on both sides of the paper) with a number. So the vertices are now labeled $1, 2, \ldots, n$. You line up the paper with the picture on the floor so that it looks like the first picture below (which illustrates what happens with an octagon or 8-gon). Now you are allowed to pick up the paper and put it back down on the floor however you want: the only rule is it has to line up with the picture on the floor. The question is, what different motions can you do here?

One thing you can do is rotate (counterclockwise or clockwise, though note that rotating clockwise by $\alpha$ is the same thing as rotating counterclockwise by $2\pi - \alpha$). In the second picture above, we have rotated the octagon counterclockwise by $\pi/4$, and in the third picture we have rotated the octagon counterclockwise by $\pi$. If we denote by $\theta$ the angle $2\pi/8$, these two rotations are then
rotations by $\theta$ and $4\theta$, respectively. It’s not hard to see that the only different rotations we can perform for this picture are counterclockwise rotations by $0, \theta, 2\theta, 3\theta, \ldots, 7\theta$.

However, there are other things we can do to the octagon. But whatever we do, note that the order of the numbers we have written will remain the same: 1 is followed by 2, 2 is followed by 3, and so on. We won’t suddenly get an octagon whose vertices are labeled $1, 3, 2, 5, 6, 7, 4, 8$ as in the picture below, because no matter what we do the numbers we’ve written on the vertices will stay exactly where they are.

However, we can reverse whether the numbers are increasing in the clockwise direction, or in the counterclockwise direction as above by flipping the octagon in one of its lines of symmetry which are drawn in the first picture below. In the second picture below, we have flipped the octagon in the line going through 1 and 5, and in the third picture we have flipped it in the line going through 4 and 8.

Note that there are precisely 8 flips we can make like this: one for every line of symmetry. Furthermore, this exhausts all of the possible motions we can perform: keeping in mind that the only labeling of the vertices we can end up with is $1, 2, 3, 4, 5, 6, 7, 8$ increasing either clockwise
or counterclockwise, it’s not hard to see that there are precisely 16 different labelings like this: 8 coming from rotating the original picture, and 8 coming from flipping the original picture.

In the more general situation of an \( n \)-gon, we let \( \theta = \frac{2\pi}{n} \), and note that the only motions we can perform are rotations \( \rho_i \) counterclockwise by \( i\theta \) for \( 0 \leq i \leq n - 1 \), and flips (reflections) in \( n \) lines of symmetry. Let’s denote the set of all \( 2n \) of these motions by \( D_n \). It turns out that \( D_n \) under composition of the motions (composing motion 1 and motion 2 means “do motion 2 then do motion 1”) is a group, and we call it a dihedral group. Let’s make this formal. Since we have been discussing regular polygons, let’s stick with \( n \geq 3 \).

**Theorem 1.** Let \( P_n \) be a regular \( n \)-gon where \( n \geq 3 \) and let \( \theta = \frac{2\pi}{n} \). The set \( D_n \) of rotations \( \rho_i \) of \( P_n \) by \( i\theta \) for \( 0 \leq i \leq n - 1 \) and reflections in the lines of symmetry of \( P_n \) is a group under composition of motions.

**Proof.** The proof of this is not difficult. Note that composition of motions is in fact a binary operation on \( D_n \): composing any two motions in \( D_n \) gives us a motion in \( D_n \). If we consider our previous set-up of a paper \( n \)-gon lining up with an \( n \)-gon drawn on the floor, we saw that every motion in \( D_n \) is such that the \( n \)-gons do line up. So if \( m_1, m_2 \in D_n \), note that after performing \( m_1m_2 \) the \( n \)-gons still line up. Furthermore, we saw above that the only motions of \( P_n \) which make these \( n \)-gons line up are the \( n \) rotations and \( n \) reflections described in the theorem. So if \( m_1, m_2 \in D_n \), \( m_1m_2 \in D_n \) as well. Therefore composing any two motions in \( D_n \) gives another motion in \( D_n \).

The fact that composition of motions is associative is left as an exercise. As for the existence of an identity element in \( D_n \), note that rotation by 0 degrees (a motion which is in \( D_n \)) composed with any other motion \( m \) either on the left or on the right is just the same as performing the motion \( m \). So \( \rho_0 \), rotation by 0 degrees, is the identity in \( D_n \).

Finally, let \( \rho_i \) denote counterclockwise rotation by \( i\theta \) where \( 0 \leq i \leq n - 1 \) as before. Clearly\(^1\), \( \rho_{n-i}\rho_i = \rho_0 \), and \( \rho_{n-i} \in D_n \) by definition, so every rotation in \( D_n \) has an inverse. Furthermore, for any reflection \( \mu \in D_n \), we have \( \mu\mu = \rho_0 \), so every reflection has an inverse in \( D_n \). Thus \( D_n \) is a group under composition of motions.

\footnote{\( \text{To be precise we should point out that if } i = 0 \text{ the inverse of } \rho_i \text{ is } \rho_i \text{ itself} \)}

Now that we have learned that \( D_n \) is a group, let’s understand better the geometry involved. We
have seen that the only motions of \( P_n \) which abide by the rules set forth in the beginning of these notes (where \( P_n \) was cut out of paper and had to line up with an \( n \)-gon drawn on the floor) are reflections and rotations. Furthermore, we have just proven that these reflections and rotations form a group under composition of motions. So composing a reflection and a rotation should yield a reflection or rotation again, which is not so intuitive, perhaps. Let’s prove that this is the case. To help us, we will set up notation as follows.

Let \( n \geq 3 \) and let \( P_n \) be the regular \( n \)-gon with vertices \((\cos j \theta, \sin j \theta)\) for \( j \in \mathbb{Z}_n \). An example of this is drawn for \( n = 8 \) below. Let \( \theta = \frac{2\pi}{n} \) and let \( \rho_i \) denote counterclockwise rotation around the origin for \( i \in \mathbb{Z}_n \). Similarly, let \( \mu_k \) denote reflection in the line going through the origin and the point \((\cos k \theta/2, \sin k \theta/2)\).

From now on all addition and subtraction is assumed to be mod \( n \). It’s clear that \( \rho_i \rho_j = \rho_{i+j} \). We’d like to understand what are \( \rho_i \mu_j \), \( \mu_j \rho_i \), and \( \mu_i \mu_j \). Let’s first consider the following question: Suppose we have a line \( \ell(\alpha) \) through the origin and \((\cos \alpha, \sin \alpha)\). Note that if we rotate this line by \( i \theta \), we end up at the line through the origin and \((\cos(\alpha + \theta), \sin(\alpha + \theta))\). Now let \( \ell(j \theta/2) \) be the line passing through the origin and \((\cos(j \theta/2), \sin(j \theta/2))\). If we reflect \( \ell(\alpha) \) through \( \ell(j \theta/2) \),
we end up at the line going through the origin and \((\cos(j\theta - \alpha), \sin(j\theta - \alpha))\). Now let’s consider the following compositions.

- \(\mu_j \rho_i\): The first thing to ask is, does this motion fix any line \(\ell(k\theta/2)\) (a line through the origin and \((\cos(k\theta/2), \sin(k\theta/2))\)? By what we just did, we see that such a line is mapped to the line going through the origin and \((\cos(j\theta - (k\theta/2 + i\theta)), \sin(j\theta - (k\theta/2 + i\theta)))\). So our question is, can we find a \(k \in \{0, 1, 2, \ldots, n - 1\}\) such that

\[
k\theta/2 = j\theta - (k\theta/2 + i\theta)\
\]

In fact, we can solve this and get \(k = j - i\). So what we have is a reflection and \(\mu_j \rho_i = \mu_{j-i}\).

- \(\rho_i \mu_j\): Again we ask does this motion fix any line \(\ell(k\theta/2)\) (a line through the origin and \((\cos(k\theta/2), \sin(k\theta/2))\)? In this case, we see that such a line is mapped to the line going through the origin and \((\cos(i\theta + (j\theta - k\theta/2)), \sin(i\theta + (j\theta - k\theta/2)))\). So our question is, can we find a \(k \in \{0, 1, 2, \ldots, n - 1\}\) such that

\[
k\theta/2 = i\theta + (j\theta - k\theta/2)\
\]

Just as in the last case, we can solve this and get \(k = j + i\). So what we have is a reflection and \(\rho_i \mu_j = \mu_{j+i}\).

- \(\mu_i \mu_j\): We proceed the same way as before: does this motion fix any line \(\ell(k\theta/2)\) (a line through the origin and \((\cos(k\theta/2), \sin(k\theta/2))\)? In this case, we see that such a line is mapped to the line going through the origin and \((\cos(i\theta - (j\theta - k\theta/2)), \sin(i\theta - (j\theta - k\theta/2)))\). So our question is, can we find a \(k \in \{0, 1, 2, \ldots, n - 1\}\) such that

\[
k\theta/2 = i\theta - (j\theta - k\theta/2) = i\theta - j\theta + k\theta/2\
\]

In fact, given arbitrary \(i\) and \(j\), we cannot solve this, because this equation is equivalent to \(0 = i\theta - j\theta\) which is false unless \(i = j\). In that case, \(\mu_i \mu_j = \mu_i \mu_i = \rho_n\) or the identity (rotation by \(2\pi\)). Therefore this motion is a rotation, not a reflection, and what we just showed is that this is a rotation which maps the line through \((\cos(k\theta/2), \sin(k\theta/2))\) to the line through
\((\cos(i\theta - j\theta + k\theta/2), \sin(i\theta - j\theta + k\theta/2))\): this is thus the rotation \(\rho_{i-j}\), since we have just added \(i\theta - j\theta\) to \(k\theta/2\). So \(\mu_i\mu_j = \rho_{i-j}\).

We end these notes by pointing out that every motion in \(D_n\) permutes the vertices which are labeled 1, 2, 3, \ldots, \(n\) in some way, and therefore \(D_n \subset S_n\), where \(S_n\) is the symmetric group on \(n\) letters for \(n \geq 3\). Since \(D_n\) is a group, it is in fact a subgroup of \(S_n\) (a proper subgroup if \(n \geq 4\), since \(|S_n| = n! > 2n = |D_n|\) for \(n \geq 4\)).

Another remark is there is a much less spectacular way to think about \(D_n\). Namely, let \(x\) denote rotation by \(2\pi/n\), and let \(y\) be any one of the reflections in \(D_n\). We know \(x^n = e\), the identity element, and that \(y^2 = e\) as well. Furthermore, you can deduce from our discussion of composition of rotations and reflections above that \(yx = x^{n-1}y\). We say \(x\) and \(y\) are generators of \(D_n\), and the equations \(x^n = y^2 = e\), \(yx = x^{n-1}y\) are relations for these generators. With this in mind the dihedral group can be thought of just as an abstract group (all geometry forgotten) of the form

\[D_n = \{x, x^2, \ldots, x^n, y, yx, yx^2, \ldots, yx^{n-1}\}\]

This presentation of the group is helpful when one considers its subgroups, for example.