Math 104: limsup/liminf and metric spaces

Note: you can assume that the metric spaces that these problems are about are nonempty

1) For any two real sequences \( \{a_n\}, \{b_n\} \), prove that \( \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n \), provided the sum on the right is not of the form \( \infty - \infty \).

2) Let \( X \) be a metric space in which every infinite subset has a limit point. Prove that \( X \) is compact.

3) Let \( S \) be a nonempty subset of \( \mathbb{R} \) and let \( c > 0 \). Prove that \( \sup \{cs \mid s \in S\} = c \cdot \sup S \).

4) Prove that in any metric space \( (X,d) \) one has \( |d(x,y) - d(x,z)| \leq d(y,z) \) for all \( x, y, z \in X \).

5) Let \( \{x_n\} \) be a bounded sequence in \( \mathbb{R} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Prove that:
   - \( \limsup f(x_n) \geq f(\limsup x_n) \)
   - \( \liminf f(x_n) \leq f(\liminf x_n) \).

6) Show that if a metric space \( (X,d) \) is compact (meaning \( X \) is compact with respect to the metric \( d \)), then there exist points \( a, b \in X \) such that \( d(a,b) = \sup \{d(x,y) : x, y \in X\} \). (Hint: You’ll need to use the fact that \( (X,d) \) is compact twice. In particular, this means that every compact metric space has an upper bound on how far away two points can be)

7) Let \( \{a_n\} \) be an enumeration of all rational numbers in \([0,1] \). Prove that \( \limsup a_n = 1 \) and \( \liminf a_n = 0 \).

8) Consider the set \( S \) of all functions from \([0,1]\) to the reals. Decide whether the definition
   \[
   d(f,g) = \int_0^1 |f-g|dx
   \]
   is a metric on \( S \).

9) Let \( \{s_n\}_{n=1}^\infty \) and \( \{t_n\}_{n=1}^\infty \) be real sequences such that for all \( n \in \mathbb{N} \), \( s_n \leq t_n \) where \( \{t_n\}_{n=1}^\infty \) is a Cauchy sequence. Prove (without using the Cauchy Criterion) that:
   \[ \limsup (s_n) \leq \liminf (t_n) \]

10) Let \( p \) be a point of a metric space \( X \) and \( r \) be a positive real number, and let \( M = \{q \in X : d(p,q) > r\} \). Prove that \( M \) is open in \( X \).
11) Suppose that \((X, d)\) is a metric space and let \(\{x_1, \ldots, x_n\}\) be a finite set of points of \(X\). Show, using only the definition of open, that the set \(X \setminus \{x_1, \ldots, x_n\}\) obtained by removing each \(x_i\) from \(X\) is open in \(X\).

12) Let \((X, d)\) be a metric space. Show that all subsets of \(X\) are open if and only if every subset of \(X\) which consists of a single point is open.

13) Let \((X, d)\) be a metric space, \(K\) a compact subset of \(X\), and \(G\) an open subset of \(X\). Prove that \(K \cap G^c\) is compact.

14) Let \(\{a_n\}\) be a sequence such that \(\liminf |a_n| = 0\). Prove that it has a subsequence \(\{b_n\}\) such that both \(b_n\) and \(\sum b_n\) converge.

15) Suppose that \(\{x_n\}\) is a sequence in a metric space \(X\) such that \(\lim x_n = a\) exists. Show that \[
\{x_n \mid n \in \mathbb{N}\} \cup \{a\}
\]
is a closed subset of \(X\).

16) Show that a continuous function from a compact metric space \((X, \rho)\) into a metric space \((Y, d)\) is uniformly continuous.

17) Let \(A\) be a compact subset of \(\mathbb{R}\) and \(B\) a closed subset of \(\mathbb{R}\). Prove the set \[
A + B = \{a + b \mid a \in A, b \in B\}
\]
is closed.

18) Let \(p\) be a point of a metric space \(X\) and \(r\) a positive real number, and let \[
M = \{q \in X \mid d(p, q) > r\}.
\]
Prove that \(M\) is open in \(X\).

19) Let \(\{s_n\}\) and \(\{t_n\}\) be bounded sequences. Prove that if for all \(n, s_n \geq 0\) and \(t_n \geq 0\), \[
\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n).
\]

20) Let \(X\) be an infinite set. For \(p\) in \(X\) and \(q\) in \(X\), define \[
d(p, q) = 1 \text{ if } p \neq q, \quad d(p, q) = 0 \text{ if } p = q.
\]
Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which
are compact?

21) Let \((X, d)\) be a compact metric space, and \(f : X \to X\) a map such that \(d(f(x), f(y)) < d(x, y)\) for all \(x \neq y\). Prove that there exists a point \(x\) such that \(f(x) = x\).

22) Let \(b(n)\) denote the number of binary digits of the natural number \(n\). Define \(a(n) = (-1)^{b(n)}\) and

\[
A(N) = \frac{\sum_{n=1}^{N} a(n)}{N}.
\]

Determine \(\limsup A(N)\) and \(\liminf A(N)\).

23) Let \(s_n\) be a bounded sequence in \(\mathbb{R}\). Let \(a = \liminf s_n\) and \(b = \limsup s_n\). Prove that \(\limsup s_n^2 = \max(a^2, b^2)\).

24) The distance from a point \(p\) in a metric space \(M\) to a non-empty subset \(S\) of \(M\) is defined to be \(d(p, S) = \inf\{d(p, s) \mid s \in S\}\). Show that \(p \notin S\) is a limit of \(S\) if and only if \(d(p, S) = 0\).

25) Prove that \(\limsup |s_n| = 0\) if and only if \(\lim s_n = 0\).

26) Let \(K\) in \(\mathbb{R}\) consist of 0 and the numbers \(1/n\), for \(n = 1, 2, 3, 4, \ldots\). Prove that \(K\) is compact directly from the definition of compactness (without using the Heine-Borel theorem).

27) Let \(\{s_n\}\) and \(\{t_n\}\) be bounded sequences. Prove that if \(s_n \geq 0\) and \(t_n \geq 0\) for all \(n\), then

\[
\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n).
\]

28) Let \((M, d)\) be a metric space.
   a) Show that the union of finitely many compact sets \(K_1, \ldots, K_n \subset M\) is compact.
   b) Suppose \(K \subset M\) is compact and \(f : K \to \mathbb{R}\) is continuous. Show that \(\{x \in K \mid f(x) = 0\}\) is compact.

29) Let \(\{S_n\}_{n \in \mathbb{N}}\) be a sequence of sets. Define:

\[
\liminf S_n = \{x \mid \text{there is an } N \text{ such that } x \in S_n \text{ for all } n > N\}
\]

\[
\limsup S_n = \{x \mid x \in S_n \text{ for infinitely many } n\}.
\]

Show that \(\liminf S_n^c = (\limsup S_n)^c\), where \(S^c\) denotes the complement of the set \(S\) in some set containing all the sets \(S_n\).

30) Let \((X, d)\) be a metric space, and let \(\{K_n\}_{n=1}^\infty\) be a countable family of non-empty compact sets such that \(K_{n+1} \subset K_n\) holds for all \(n\). Let \(U \subset X\) be an open set such that \(\bigcap_{n=1}^\infty K_n \subset U\). Prove that there exists a
positive integer $N$ such that $K_N \subset U$.

31) Let $(a_n)_{n \geq 1}$ be a sequence of real numbers with $a_n > 0$ for each $n \geq 1$. Consider the following:

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} a_k$$

Prove

$$\liminf_{n \to \infty} a_n \leq \liminf_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} a_n$$

Conclude if $\lim_{n \to \infty} a_n$ exists, then $\lim_{n \to \infty} \sigma_n$ exists and $\lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} a_n$.

32) Let $(S,d_S),(T,d_T)$ be two metric spaces and $f : S \to T$ be a continuous function. Show that, if $K \subset S$ is compact, then $f(K)$ is also compact.

33) True or False? If $f : X \to Y$ is continuous, $E \subset X$ is closed and $X$ is compact, then $f(E) \subset Y$ is closed.

34) a) Define an open cover and a compact set.

b) Show explicitly that an open interval $(a,b)$ in $\mathbb{R}$ is not compact.

35) Let $\alpha = \liminf(1/s_n)$ and $\beta = \limsup(s_n)$ with $1/2 < s_n < 2$ for all $n$. Show that $\alpha \beta = 1$.

36) Let $\{a_n\}$ be a bounded sequence of real numbers. Define $\{s_n\}$ as $s_n = (a_1 + \cdots + a_n)/n$ for all $n$. Prove that $\liminf a_n \leq \liminf s_n \leq \limsup s_n \leq \limsup a_n$.

37) Let $X$ be a metric space and define the diameter of a metric space $S$ to be $diam(S) = \sup\{d(x,y) \mid x,y \in S\}$. Suppose $K_1, K_2, \ldots$ are nonempty compact subsets of $X$ such that $K_n$ is contained in $K_{n+1}$ for all $n$. Let $K$ be the intersection of all these $K_i$. Prove that $diam(K) = \lim diam(K_n)$.

38) Show that the following definition of $\limsup x_n$ is equivalent to the one that we have seen in class:

$$\limsup x_n = \inf_{n \geq 1} \sup \{k \geq nx_n\}.$$  

Can you think of a similar definition of $\liminf$?

39) Show that $\limsup x_n = -\liminf(-x_n)$ and $\liminf x_n = -\limsup(-x_n)$.

40) Show that the set of integers, as well as any finite subset of $\mathbb{R}$ is a closed but not a perfect set.