Math 286 G1 - Midterm 2 Practice

Clearly, it is too long for a single exam. Take the first 7 for timed trial.

1. For the unforced oscillation equation

\[ m x''(t) + c x'(t) + k x(t) = 0. \]

Derive the criteria that distinguishes under-, critical-, over- damped cases. Then write down the formula for general solutions for each case.

2. Show that, one particular solution to the forced oscillation with damping equation

\[ m x''(t) + c x'(t) + k x(t) = F_0 \cos(\omega t) \]

is

\[ x_p(t) = \frac{F_0}{\sqrt{(k - m^2 \omega^2)^2 + (c \omega)^2}} \cos(t - \alpha), \quad \text{for some } \alpha. \]

3. Consider the boundary value problem

\[
\begin{cases}
  y'' + \lambda y = 0, \\
  y'(\pi) = y'(-\pi) = 0.
\end{cases}
\]

Find all \( \lambda \) so that there is a non-zero (trivial) solution to the system, also write down the corresponding solutions.

( P.S., These \( \lambda \)s are called eigen-values and the solutions are eigen-functions. )

4. Using the Fourier series method to solve the second order inhomogeneous differential equation

\[ y'' + 2y = \cos^2(3x) - \sin^2(3x) + 2 \sin(x) \]

5. Find the Fourier Series of the periodic equation

\[ f(x) = |\cos x| \]

6. Find the solution to the wave equation with boundary data:

\[
\begin{cases}
  y_{tt}(x, t) = 16 y_{xx}(x, t) \\
  y(0, t) = y(2\pi, t) = 0 \\
  y(x, 0) = \sin(x), \quad y_t(x, 0) = \cos(x)
\end{cases}
\]
7. Find the solution to the heat equation with boundary data:

\[
\begin{cases}
y_t(x, t) = 16y_{xx}(x, t) \\
y(0, t) = y\left(\frac{x}{2}, t\right) = 0, \quad y(x, 0) = 5
\end{cases}
\]

8. Determine that the following functions are a) odd only, b) even only, c) both even and odd, d) neither even nor odd.

(a) \( f(x) = 0 \)
(b) \( f(x) = (\sin(2x))^2 - (\cos(x))^3 \)
(c) \( f(x) = \ln(|x^3 - 3x|) \)
(d) \( f(x) = e^{-x^2-x} \)
(e) \( f(x) = 5 + \sum_{i=1}^{\infty} \frac{1}{4i^2} \cos(2ix) \)

9. Use the variation of parameter method to solve the following equation

\[ y'' + y' - 6y = 12x + 10 \]

10. Finally, a Harmonic equation

\[ y_{rr} + \frac{1}{r} y_r + \frac{1}{r^2} y_{\theta\theta} = 0 \quad 0 \leq r \leq 1, \quad \theta \in [0, 2\pi) \]

\[ y(1, \theta) = \begin{cases} 
1 & \text{for } \theta \in (0, \pi) \\
-1 & \text{for } \theta \in (\pi, 2\pi)
\end{cases} \]