1. For the first order linear equation

\[ y' + P(x)y = Q(x) \]

(a) Write down the formula for the integral factor, which is \( \rho \) in my lecture note:

\[ H = e^{\int P(x) dx} \]

(b) Using the integral factor of above, solve the following equation with initial data

\[ y' + 2xy = 2x, \quad y(0) = 5 \]

\[ H = e^{x^2}, \quad y = 1 + Ce^{-x^2}, \quad y = 1 + 4e^{-x^2} \]

2. For the second order linear equation

\[ y'' + 7y' + 10y = 0 \]

(a) Find two solutions of the differential equation above and use the Wronskian method to show that they are linearly independent.

\[ y_1 = e^{-5x}, \quad y_2 = e^{-2x}, \quad W = 3e^{-7x} \neq 0 \]

(b) Find the particular solution which satisfies the initial data

\[ y(0) = 2, \quad y'(0) = -1 \]

\[ y(x) = -e^{-5x} + 3e^{-2x} \]

3. Using the substitution method to solve the Bernoulli type equation (Section 1.6: 22)

\[ x^2y' + 2xy = 5y^4 \]

\[ v = y^{-3}, \quad y = \left( \frac{7x}{15 + Cx^7} \right)^{1/3} \]

4. For the linear differential equation

\[ y''' + 4y'' + 4y' = 0 \]

(a) Write the characteristic equation and solve

\[ r^3 + 4r^2 + 4r = 0, \quad r_1 = 0, r_2 = r_3 = -2 \]
(b) Find the general solution of the differential equation

$$y_c = C_1 + C_2e^{-2x} + C_3xe^{-2x}$$

(c) Find a particular solution, where the initial data is $y(0) = 2$, $y'(0) = 1$, $y''(0) = -3$

$$y = \frac{9}{4} - \frac{1}{4}e^{-2x} + \frac{1}{2}xe^{-2x}$$

5. Given the non-homogeneous equation

$$y'' - 4y' - 12y = 2\sin(3x)$$

(a) Find the solution, $y_c$, to the homogeneous counterpart of this equation

$$y_c = C_1e^{6x} + C_2e^{-2x}$$

(b) Find a particular solution, $y_p$, to the original equation using the method of variation of parameters.

$$y_p = \frac{8}{195}\cos(3x) - \frac{14}{195}\sin(3x)$$

6. For the un-forced oscillation model

$$mx'' + cx' + kx = 0$$

(a) Find a criteria, with detail, to distinguish the under-damping, critical-damping and over-damping cases. Write the general solution, $y_c$, for each case.

(b) Write the solution to the inhomogeneous case, when the right hand side is replaced by $F_0\cos(\omega t)$ knowing $c \neq 0$

SEE THE BOOK

7. For the system with parameter $\lambda$:

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \quad y(-\pi) = 0 \end{cases}$$

(a) Show that, for negative $\lambda$, the system has no non-trivial solution, i.e. there is no negative eigenvalues.

(b) Show that $\lambda = 144$ is an eigenvalue for the system. Find the eigenfunction for this $\lambda$. SEE THE BOOK

8. Determine that the following functions are odd only / even only / both odd and even / neither odd nor even .

(a) $f(x) = 0$ Both
(b) \( g(x) = \sin(x) + \cos(-2x) \) Neither
(c) \( h(x) = x^4 - \frac{1}{x^2} + \tan(x) \quad x \neq \frac{n\pi}{2} \) Neither
(d) \( k(x) = (\cos(-2x))^3 + (\sin(3x))^2 \) Even Only
(e) \( m(x) = e^{-2x^2} \) Even Only

9. Using the Fourier series technique to solve the second order in-homogeneous linear differential equation.

\[ y'' + 2y' - 2y = \cos^2(x) - \sin^2(x) - \sin(x) \quad x \in (0, \pi) \]

\[ y_p = \frac{2}{13} \cos(x) + \frac{3}{13} \sin(x) - \frac{3}{26} \cos(2x) + \frac{1}{13} \sin(2x) \]

10. Solve the heat equation by Fourier series.

\[ y_t(x, t) = 16 y_{xx}(x, t), \quad t \in [0, \infty) \quad x \in [0, \frac{\pi}{2}] \]

\[ y(0, t) = y\left(\frac{\pi}{2}, t\right) = 0, \quad y(x, 0) = 5 \]

\[ y = \sum_{n \text{ odd}} \frac{20}{n \pi} e^{-6n^2\pi t} \sin(2nx) \]

11. Solve the wave equation by Fourier series

\[ y_{tt}(x, t) = 4 y_{xx}(x, t), \quad t \in [0, \infty) \quad x \in [0, 1] \]

\[ y(0, t) = y(1, t) = 0, \quad y(x, 0) = \sin^3(\pi x), \quad y_t(x, 0) = \sin(\pi x) \]

\[ y = \frac{3}{4} \sin(\pi x) \cos(2\pi t) - \frac{1}{4} \sin(3\pi x) \cos(6\pi t) + \frac{1}{2\pi} \sin(\pi x) \sin(2\pi t) \]

12. For the matrix

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]

(a) Find \( A^2 \)

\[ A^2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & 0 \end{pmatrix} \]

(b) Find the inverse of \( A \) if possible

Does not exist
(c) Find all the three eigenvalues of this matrix and their corresponding eigenvectors.
\[ \lambda_1 = 2, \quad v_1 = (2, 2, 1)^T; \quad \lambda_2 = 1, \quad v_2 = (1, 0, 1)^T; \quad \lambda_3 = 0, \quad v_3 = (0, 0, 1)^T \]

13. Solve the in-homogeneous system of equations with initial data
\[ \vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{2t} \\ e^{-t} \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ \vec{x} = \begin{pmatrix} \frac{2}{3}e^{2t} + \frac{1}{4}e^t - \frac{1}{2}te^{-t} + \frac{1}{12}e^{-t} \\ \frac{1}{3}e^{2t} + \frac{1}{4}e^t + \frac{1}{2}te^{-t} - \frac{7}{12}e^{-t} \end{pmatrix} \]

14. Find the exponential of the matrix A below, i.e. \( e^{At} \)
\[ A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \]
\[ e^{At} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 3t & 6t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & 3te^{2t} & 6t^2e^{2t} \\ 0 & e^{2t} & 4te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \]

15. Find the Laplace Transform of the followings
(a) \( f(t) = t^2 + e^{3t} - 5 \cos(5x) \)
\[ F(s) = \frac{2}{s^3} + \frac{1}{s-3} - \frac{5s}{s^2+25} \]
(b) \( f(t) = t^2 \cos(2t) \)
\[ F(s) = \frac{2s^3 - 24s}{(s^2+4)^3} \]
(c) \( f^{(3)}(t) \)
\[ s^3F(s) - sf(0) - sf'(0) - f''(0) \]

16. Find the inverse Laplace Transform of the followings
(a) \( F(s) = \frac{1}{(s+1)(s-1)} \)
\[ f(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^t \]
(b) \( F(s) = \frac{2}{s(s^2+4)} \)
\[ f(t) = \frac{1}{2} - \frac{1}{2} \cos(2t) \]

Table of Laplace Transform
\[
f(t) \quad F(s)
\]

| \(f^k e^{at}\) | \(\frac{k!}{(s-a)^{k+1}}\) |
| \(\sin(at)\) | \(\frac{a}{s^2+a^2}\) |
| \(\cos(at)\) | \(\frac{s}{s^2+a^2}\) |
| \(tf(t)\) | \(-F'(s)\) |
| \(f'(t)\) | \(sF(s) - f(0)\) |
| \(\frac{1}{s} \int_0^t f(\tau)d\tau\) | \(\frac{1}{s}F(s)\) |

Other Formulas

1. \(\sin(2x) = 2 \sin(x) \cos(x)\) \(\cos(2x) = \cos^2(x) - \sin^2(x)\)
   \(\sin(3x) = 3 \sin(x) - 4 \sin^3(x)\) \(\cos(3x) = 4 \cos^3(x) - 3 \cos(x)\)

2. \(\sin(\alpha) \cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))\)
   \(\sin(\alpha) \sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))\)
   \(\cos(\alpha) \cos(\beta) = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))\)

3. The Fourier series for the \(2L\)-periodic function \(f\), with value 1 on \((0, L)\), \(-1\) on \((-L, 0)\) and 0 at 0, \(L\) and \(-L\) is
   \[
f(x) = \frac{4}{\pi} \sum_{j \text{ odd}} \frac{1}{j} \sin\left(\frac{j\pi}{L}x\right)
   \]

4. The Fourier series for the \(2L\)-periodic function \(f\), with formula \(x\) on \([0, L)\) and \(2L-x\) on \([L, 2L)\) is
   \[
f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{i\pi}{L}x\right)
   \]

5. The Fourier series for the \(2L\)-periodic function \(f\), with formula \(x\) on \((-L, L)\) and 0 at \(L\) and \(-L\) is
   \[
f(x) = \frac{2L}{\pi} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{n} \sin\left(\frac{i\pi}{L}x\right)
   \]