1. For the first order linear equation

\[ y' + P(x)y = Q(x) \]

(a) Write down the formula for the integral factor,
(b) Using the integral factor of above, solve the following equation with initial data

\[ y' + 2xy = 2x, \quad y(0) = 5 \]

2. For the second order linear equation

\[ y'' + 7y' + 10y = 0 \]

(a) Find two solutions of the differential equation above and use the Wronskian method to show that they are linearly independent.
(b) Find the particular solution which satisfies the initial data

\[ y(0) = 2, \quad y'(0) = -1 \]

3. Using the substitution method to solve the Bernoulli type equation (Section 1.6: 22)

\[ x^2y' + 2xy = 5y^4 \]

4. For the linear differential equation

\[ y''' + 4y'' + 4y' = 0 \]

(a) Write the characteristic equation and solve
(b) Find the general solution of the differential equation
(c) Find a particular solution, where the initial data is \( y(0) = 2, y'(0) = 1, y''(0) = -3 \)

5. Given the non-homogeneous equation

\[ y'' - 4y' - 12y = 2\sin(3x) \]

(a) Find the solution, \( y_c \), to the homogeneous counterpart of this equation
(b) Find a particular solution, \( y_p \), to the original equation using the method of variation of parameters.
6. For the un-forced oscillation model

\[ mx'' + cx' + kx = 0 \]

(a) Find a criteria, with detail, to distinguish the under-damping, critical-damping and over-damping cases. Write the general solution, \( y_c \), for each case.

(b) Write the solution to the inhomogeneous case, when the right hand side is replaced by \( F_0 \cos(\omega t) \) knowing \( c \neq 0 \)

7. For the system with parameter \( \lambda \):

\[
\begin{cases}
y'' + \lambda y = 0 \\
y(0) = 0 \quad y(-\pi) = 0
\end{cases}
\]

(a) Show that, for negative \( \lambda \), the system has no non-trivial solution, i.e. there is no negative eigenvalues.

(b) Show that \( \lambda = 144 \) is an eigenvalue for the system. Find the eigenfunction for this \( \lambda \).

8. Determine that the following functions are odd only / even only / both odd and even / neither odd nor even .

(a) \( f(x) = 0 \)

(b) \( g(x) = \sin(x) + \cos(-2x) \)

(c) \( h(x) = x^4 - \frac{1}{x} + \tan(x) \quad x \neq \frac{n\pi}{2} \)

(d) \( k(x) = (\cos(-2x))^3 + (\sin(3x))^2 \)

(e) \( m(x) = e^{-2x^2} \)

9. Using the Fourier series technique to solve the second order in-homogeneous linear differential equation.

\[ y'' + 2y' - 2y = \cos^2(x) - \sin^2(x) - \sin(x) \quad x \in (0, \pi) \]

10. Solve the heat equation by Fourier series.

\[ y_t(x, t) = 16 y_{xx}(x, t), \quad t \in [0, \infty) \quad x \in [0, \pi/2] \]

\[ y(0, t) = y(\pi/2, t) = 0, \quad y(x, 0) = 5 \]

11. Solve the wave equation by Fourier series

\[ y_{tt}(x, t) = 4 y_{xx}(x, t), \quad t \in [0, \infty) \quad x \in [0, 1] \]

\[ y(0, t) = y(1, t) = 0, \quad y(x, 0) = \sin^3(\pi x), \quad y_t(x, 0) = \sin(\pi x) \]
12. For the matrix
\[ A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
1 & 0 & 0
\end{pmatrix} \]

(a) Find \( A^2 \)
(b) Find the inverse of \( A \) if possible
(c) Find all the three eigenvalues of this matrix and their corresponding eigenvectors.

13. Solve the in-homogeneous system of equations with initial data
\[ \ddot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ e^{-t} \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

14. Find the exponential of the matrix \( A \) below, i.e. \( e^{At} \)
\[ A = \begin{pmatrix}
2 & 3 & 0 \\
0 & 2 & 4 \\
0 & 0 & 2
\end{pmatrix} \]

15. Find the Laplace Transform of the followings
(a) \( f(t) = t^2 + e^{3t} - 5 \cos(5x) \)
(b) \( f(t) = t^2 \cos(2t) \)
(c) \( f^{(3)}(t) \)

16. Find the inverse Laplace Transform of the followings
(a) \( F(s) = \frac{1}{(s+1)(s-1)} \)
(b) \( F(s) = \frac{2}{s(s^2+4)} \)
Table of Laplace Transform

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^k e^{at}$</td>
<td>$\frac{k!}{(s-a)^{k+1}}$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$\frac{a}{s^2+a^2}$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2+a^2}$</td>
</tr>
<tr>
<td>$tf(t)$</td>
<td>$-F'(s)$</td>
</tr>
<tr>
<td>$f'(t)$</td>
<td>$sF'(s)$ - $f(0)$</td>
</tr>
<tr>
<td>$\frac{1}{t}f(t)$</td>
<td>$\int_s^\infty F(\sigma)d\sigma$</td>
</tr>
<tr>
<td>$\int_0^t f(\tau)d\tau$</td>
<td>$\frac{1}{2}F(s)$</td>
</tr>
</tbody>
</table>

Other Formulas

1. $\sin(2x) = 2\sin(x)\cos(x)$  $\cos(2x) = \cos^2(x) - \sin^2(x)$
   $\sin(3x) = 3\sin(x) - 4\sin^3(x)$  $\cos(3x) = 4\cos^3(x) - 3\cos(x)$

2. $\sin(\alpha) \cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$
   $\sin(\alpha) \sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$
   $\cos(\alpha) \cos(\beta) = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$

3. The Fourier series for the $2L$-periodic function $f$, with value 1 on $(0, L)$, $-1$ on $(-L, 0)$ and 0 at 0, L and $-L$ is
   \[ f(x) = \frac{4}{\pi} \sum_{j \text{ odd}} \frac{1}{j} \sin(\frac{j\pi}{L} x) \]

4. The Fourier series for the $2L$-periodic function $f$, with formula $x$ on $[0, L)$ and $2L - x$ on $[L, 2L)$ is
   \[ f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{i \text{ odd}} \frac{1}{i^2} \cos(\frac{i\pi}{L} x) \]

5. The Fourier series for the $2L$-periodic function $f$, with formula $x$ on $(-L, L)$ and 0 at $L$ and $-L$ is
   \[ f(x) = \frac{2L}{\pi} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j} \sin(\frac{j\pi}{L} x) \]