Jerome Loeb Undergraduate Lecture

The Power and Beauty of Undergraduate Mathematics: Three case studies

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Case 1. Matrices and probability

For the "love-life" experiment, write

$$p_k = \text{probability of being Partnered after } k \text{ die rolls}$$
$$s_k = \text{probability of being Single after } k \text{ die rolls}$$

Then

$$p_k = \frac{4}{6}p_{k-1} + \frac{3}{6}s_{k-1}$$
$$s_k = \frac{2}{6}p_{k-1} + \frac{3}{6}s_{k-1}$$

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Then

$$\begin{aligned} p_k &= \frac{4}{6} p_{k-1} + \frac{3}{6} s_{k-1} \\ s_k &= \frac{2}{6} p_{k-1} + \frac{3}{6} s_{k-1} \end{aligned}$$

That is, we have a "Markov chain" $\boxed{x_k = T x_{k-1}}$ where

$$T = \text{transition matrix} = \begin{pmatrix} 4/6 & 3/6 \\ 2/6 & 3/6 \end{pmatrix} \quad \text{and} \quad x_k = \begin{pmatrix} p_k \\ s_k \end{pmatrix}$$

Example, cont.

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If $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (starting out Partnered) then . . .

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$$x_1 = Tx_0 = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/6 \\ 2/6 \end{pmatrix}$$

$$x_2 = Tx_1 = \begin{pmatrix} 22/36 \\ 14/36 \end{pmatrix}$$

$$x_3 = Tx_2 = \begin{pmatrix} 130/216 \\ 86/216 \end{pmatrix} \approx \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = \lim_{k\to\infty} x_k$$

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$$\begin{aligned} x_1 &= Tx_0 = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/6 \\ 2/6 \end{pmatrix} \\ x_2 &= Tx_1 = \begin{pmatrix} 22/36 \\ 14/36 \end{pmatrix} \\ x_3 &= Tx_2 = \begin{pmatrix} 130/216 \\ 86/216 \end{pmatrix} \approx \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = \lim_{k\to\infty} x_k \end{aligned}$$

In the limit, one is Partnered 3/5 of the time and Single 2/5 of the time. Same is true if start out Single.

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$Tu_1 = \lambda_1 u_1$	$u_1 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$	$\lambda_1 = 1$
$Tu_2 = \lambda_2 u_2$	$u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\lambda_2 = 1/6$

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Diagonalize T with

$$U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 3/5 & 1 \\ 2/5 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

so that

$$T = UDU^{-1}$$

Underlying matrix theory, cont.

Investigate

$$T = UDU^{-1}$$

$$T^2 = UDU^{-1}\, UDU^{-1} = UD^2U^{-1}$$

$$T^k = UD^kU^{-1} = U\begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix}U^{-1}, \quad (\lambda_1 = 1, \lambda_2 = 1/6)$$

$$\rightarrow U\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}U^{-1} = \begin{pmatrix}3/5 & 3/5\\ 2/5 & 2/5\end{pmatrix}$$

as $k \rightarrow \infty$.

Underlying matrix theory, cont.

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as $k \to \infty$. So

$$x_k = T^k x_0 = T^k\begin{pmatrix}p_0\\ s_0\end{pmatrix} \to \begin{pmatrix}3/5 & 3/5\\ 2/5 & 2/5\end{pmatrix}\begin{pmatrix}p_0\\ s_0\end{pmatrix} = \begin{pmatrix}3/5\\ 2/5\end{pmatrix}$$

since $p_0 + s_0 = 1$. Thus the long run probabilities are $3/5, 2/5$.

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since $p_0 + s_0 = 1$. Thus the long run probabilities are $3/5, 2/5$. *Challenge.* Find the long run probabilities for the general 2-state Markov chain with $T = \begin{pmatrix}1-\alpha & \beta\\ \alpha & 1-\beta\end{pmatrix}, 0 < \alpha, \beta < 1.$

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The Power of Undergraduate Mathematics. Convergence to long run probabilities is explained by Perron's Theorem and matrix diagonalization using eigenvalues and eigenvectors. The long run probabilities are entries in the eigenvector with eigenvalue $\lambda_1 = 1$.

Case 2. Growth Models in Applied Mathematics: Weed Population

Canada thistle

Let $x_k = \begin{pmatrix} \# \text{ seeds} \\ \# \text{ rosettes} \\ \# \text{ plants} \end{pmatrix}$ after year k.

Then $\boxed{x_k = Gx_{k-1}}$ where

$$G = \text{growth matrix} = \begin{pmatrix} g_{11} & 0 & g_{13} \\ g_{21} & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{pmatrix}$$

(each $g_{ij} > 0$ determined experimentally).

Weed Population Growth, cont.

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Perron's Theorem implies the dominant eigenvalue of G is $\lambda_1 > 0$, so that $x_k = G^k x_0 \approx \lambda_1^k(\text{const})u_1$. That is,

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The Power of Undergraduate Mathematics: evaluate $\frac{\partial \lambda_1}{\partial g_{ij}}$ to determine which biocontrol agent affects λ_1 the most. The validity of this "sensitivity analysis" ultimately depends on the Implicit Function Theorem.

Case 3. Matrices and Derivatives — Basic Analogies

Matrix multiplication
$u \mapsto Au$

is Linear
$A(c_1 u_1 + c_2 u_2)$
$= c_1 A u_1 + c_2 A u_2$

and Invertible
$Au = b \Leftrightarrow u = A^{-1} b.$

Domain is all vectors u in the *finite* dimensional set $\mathbb{R}^n$.

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Second differentiation
$f \mapsto f''$

is Linear
$(c_1 f_1 + c_2 f_2)'' = c_1 f_1'' + c_2 f_2''$

and Invertible
$f'' = g \Leftrightarrow f(x) = f(0) + x f'(0) + \int_0^x \int_0^y g(z)\, dz dy.$

Domain is all functions f in the *infinite* dimensional set $\{f : f'' \text{ exists}\}$.

Matrices and Derivatives — Spectral Analogy

Eigenvectors u_k and eigenvalues λ_k of a symmetric matrix A tell us everything about how A acts on vectors.

The eigenvectors span $\mathbb{R}^n$ and A acts by multiplication on the eigenspaces:

$$A(c_1 u_1 + \cdots + c_n u_n) = c_1 \lambda_1 u_1 + \cdots + c_n \lambda_n u_n.$$

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Eigenfunctions $f_k(x) = \sin kx$ and eigenvalues $\lambda_k = -k^2$ ($f_k'' = \lambda_k f_k$) of the second derivative operator on $0 < x < \pi$ with zero boundary conditions tell us everything about how the second derivative acts on nice functions.

Indeed the eigenfunctions span all reasonable functions (by Fourier series), and the second derivative acts by multiplication on the eigenspaces:

$$(c_1 f_1 + c_2 f_2 + \cdots)'' \\ = c_1 \lambda_1 f_1 + c_2 \lambda_2 f_2 + \cdots .$$

Matrices and Derivatives — Evolutionary Analogy

$Au_k = \lambda_k u_k$
The differential equation

$$\frac{du}{dt} = Au$$

has dominant solution

$$u(t) = e^{\lambda_1 t} u_1$$

as $t \to \infty$, where λ_1 is the largest eigenvalue.

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$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

with $f = 0$ at $x = 0, x = \pi$, has dominant solution

$$f(x,t) = e^{\lambda_1 t} f_1(x)$$

as $t \to \infty$, where $\lambda_1 = -1^2$ is the largest eigenvalue.

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The Beauty of Undergraduate Mathematics: Here we glimpse a single "spectral theory" that covers both matrices and unbounded linear operators such as $f \mapsto f''$ or more generally Schrödinger operators in quantum mechanics ($-f'' + Vf = Ef$, where $E =$ eigenvalue or energy level).

Exhortation and References

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Seek to exploit its **Power**, and to appreciate its **Beauty**.

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References

"The many proofs and applications of Perron's theorem", by C. R. MacCluer, SIAM Review 42:487–498, 2000.

Finite Markov Chains, by J. G. Kemeny and J. L. Snell, 1976.

Matrix Population Models, 2nd ed., by H. Caswell, 2001.

"When does it make sense to target weed seeds?" by A. S. Davis, Weed Science 51, to appear.