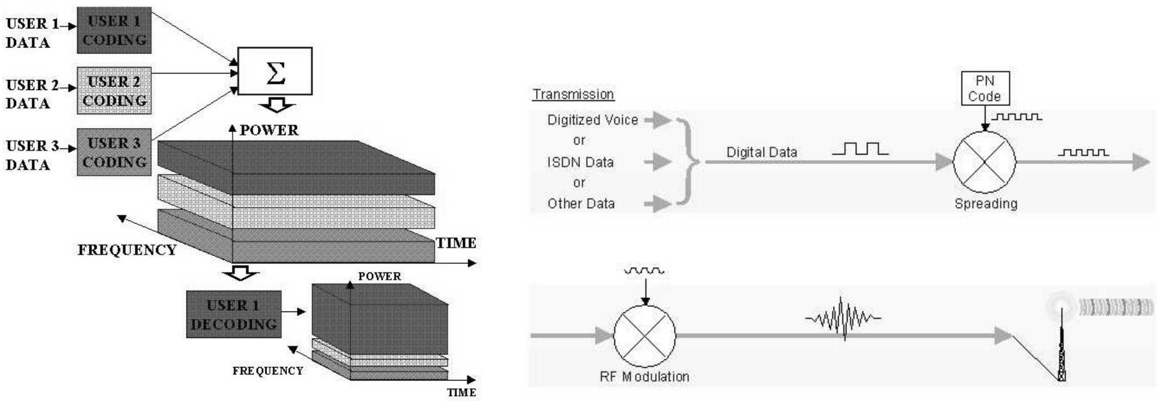
Frames in CDMA Communication Systems: Tight Frames and Their Fundamental Inequality

Ka Lung Law and Behzad Sharif

I. Brief Overview of CDMA Systems

For radio systems there are two resources, frequency and time. Division by frequency (or time), so that each pair of communicators is allocated part of the spectrum for all of the time (or all of the spectrum for part of the time), results in Frequency (or Time) Division Multiple Access (FDMA or TDMA). In Code Division Multiple Access (CDMA), every communicator will be allocated the entire spectrum all of the time (Fig. 1). CDMA has applications in wireless cellular communication as well as navigation (e.g., GPS) systems.

CDMA uses codes (here referred to as signature sequences) to identify connections and is an interference limited multiple access system. Because all users transmit on the same frequency, internal interference generated by the system is the most significant factor in determining system capacity and communication quality (e.g., quality of voice in mobile phones). In what follows, we will review frame-theoretic results associated with the problem of designing optimal, i.e. minimum interference, signature sequences in CDMA communication systems.



(a) (b)

Fig. 1. (a) Demonstration of code-division multiplexing (b) Transmitter structure of a CDMA system.

II. Frame Theory in the Uniform-Norm CDMA Problem [1], [2]

In a CDMA system, there are m users who share the available spectrum. The sharing is achieved by "scrambling" m-dimensional user vectors into smaller, n-dimensional vectors. In terms of frame theory, this scrambling corresponds to the application of a synthesis operator corresponding to m distinct n-dimensional *signature sequences* $\{\varphi_i\}_{i=1}^{m}$ each of norm $\|\varphi_i\| = a_i$. Noise-corrupted versions of these synthesized vectors arrive at a receiver, where the signature vectors are used to help extract the original user vectors (sequence of data symbols). In this section we consider the uniform-norm problem. Hence, all φ_i are of equal norm, $a_i = \sqrt{n}, i = 1, \cdots m$.

The signature sequence assigned to user i is a binary (± 1 valued) vector of length n,

$$\varphi_i = [x_1(i), x_2(i), ..., x_n(i)],$$

The binary ($\{0, 1\}$ valued) data sequence, $\underline{B}(i) = [..., B_{-1}(i), B_0(i), B_1(i), ...]$ of user i is then expanded to a binary sequence at n times the original data rate by using the original data symbols to control the polarity of the signature sequence φ_i. Each component of the expanded binary data symbol

$$B_k(i)\varphi_i = [B_k(i)x_1(i), B_k(i)x_2(i), ..., B_k(i)x_n(i)]$$

is called a *chip*. In this manner, one creates the binary (± 1 valued) sequence

$$\underline{B}(i) \otimes \varphi_i = [..., B_{-1}(i)\varphi_i, B_0(i)\varphi_i, B_1(i)\varphi_i, ...]$$

that forms the input to the modulator (transmitter) of user i. The data from all the users are added together and transmitted simultaneously. This addition happens digitally at the chip rate.

Example.

Step	Encode User 1	Encode User 2
1	$\varphi_1 = (1, -1)$, $d_1 = (1, 0, 1, 1) \Rightarrow B_1 = (1, -1, 1, 1)$	$\varphi_2 = (1, 1)$, $d_2 = (0, 0, 1, 1) \Rightarrow B_2 = (-1, -1, 1, 1)$
2	$e_1 = ((1, -1), (-1, 1), (1, -1), (1, -1))$	$e_2 = ((-1, -1), (-1, -1), (1, 1), (1, 1))$

Because both signals are transmitted at the same time into the air, they add to produce the raw signal:
$r = e_1 + e_2 = ((1-1), (-1, 1), (1, -1), (1, -1)) + ((-1, -1), (-1, -1), (1, 1), (1, 1)) = ((0, -2), (-2, 0), (2, 0), (2, 0))$

Step	Decode User 1	Decode User 2
1	$\varphi_1 = (1, -1), r = ((0, -2), (-2, 0), (2, 0), (2, 0))$	$\varphi_2 = (1, 1), r = ((0, -2), (-2, 0), (2, 0), (2, 0))$
2	$S_1 = ((0, -2), (-2, 0), (2, 0), (2, 0)).(1, -1)$	$S_2 = ((0, -2), (-2, 0), (2, 0), (2, 0)).(1, 1)$
3	$S_1 = ((0 + 2), (-2 + 0), (2 + 0), (2 + 0))$	$S_2 = ((0 - 2), (-2 + 0), (2 + 0), (2 + 0))$
4	$S_1 = (2, -2, 2, 2) \Rightarrow \hat{d}_1 = (1, 0, 1, 1)$	$S_2 = (-2, -2, 2, 2) \Rightarrow \hat{d}_2 = (0, 0, 1, 1)$

With the usual assumption of additive white Gaussian noise, this implies that the received (demodulated) signal for the k-th data symbol interval can be written as the n-chip sequence

$$r_k = \sum_{j=1}^{m} B_k(j)\varphi_j + \eta_k \tag{1}$$

where the n-chip noise sequence $\eta_k = [\eta_{k1}, \eta_{k2}, ..., \eta_{kn}]$, is vector of random variables, each with mean 0 and variance $1/g$, where g is the signal-to-noise ratio.

In the CDMA receiver, the sequence r_k is further processed separately for each user. For user i, the inner product of its signature sequence φ_i with r_k provides the "detection statistic $S_k(i)$, i.e.,

$$S_k(i) = \langle r_k, \varphi_i \rangle = \sum_{l=1}^{n} r_{kl} x_l(i)$$

where $r_k = [r_{k1}, r_{k2}, ..., r_{kn}]$. With the help of (1) and the fact that

$$\langle \varphi_j, \varphi_j \rangle = n \tag{2}$$

for all j, the data symbol detection statistic for user i becomes

$$S_k(i) = nB_k(i) + \sum_{j=1, j\neq i}^{m} B_k(j)\langle \varphi_j, \varphi_i \rangle + h_k \tag{3}$$

where

$$h_k(i) = \langle \eta_k, \varphi_i \rangle = \sum_{l=1}^{n} \eta_{kl} x_l(i)$$

is a Gaussian random variable with mean 0 and variance n/g that is independent of the data symbols. Because the data symbols of the m users are themselves statistically independent and each has mean 0 and variance 1, the sum

$$\zeta_k(i) = \sum_{j=1, j\neq i}^{m} B_k(j)\langle \varphi_j, \varphi_i \rangle$$

in (3), which represents the *interuser interference* experienced by user i, has mean 0 and variance

$$\sigma_i^2 = \sum_{j=1, j\neq i}^{m} |\langle \varphi_j, \varphi_i \rangle|^2 = \sum_{j=1}^{m} |\langle \varphi_j, \varphi_i \rangle|^2 - n^2 \qquad \text{by (2)}$$

leading to the total interuser interference: $\sigma_{tot}^2 = \sum_{i,j=1}^{m} |\langle \varphi_j, \varphi_i \rangle|^2 - mn^2$

Definition 1: Consider the Cartesian product $S(\{a_i\}_{i=1}^m) = S(a_1) \times \cdots \times S(a_m)$ where $S(a_i) = \{\varphi \in \mathbb{H}^n : \|\varphi\| = a_i\}$. For any $\{\varphi_m\} \in S(\{a_i\}_{i=1}^m)$, the frame potential is defined as

$$FP(\{\varphi_i\}_{i=1}^m) = \sum_{i,j=1}^{m} |\langle \varphi_j, \varphi_i \rangle|^2.$$

From the Def. 1 (with all a_i's equal to $\sqrt{n}$), it follows that: $\sigma_{tot}^2 = FP(\{\varphi_i\}_{i=1}^m) - mn^2$.

The goal is to minimize the total interference. It should be clear that no interference is possible if and only if all φ_i are orthogonal, and in turn, this is possible only if $m \leq n$, or, when φ_i either form an orthogonal set or an orthonormal basis. When $m > n$, $FP - mn^2 \geq FP - m^2 n$. The result, which is known as the Welch bound within the engineering community (due to L.R. Welch in 1974), is

$$\sum_{i,j=1}^{m} |\langle \varphi_j, \varphi_i \rangle|^2 \geq mn^2$$

with equality if and only if the $m \times n$ matrix Φ^* whose rows are φ_i^*, has orthogonal columns of norm $\sqrt{n}$. If we normalize each φ_i's to be unit norm, this is equivalent to the following theorem by Bendetto and Fickus [3].

Theorem 1: Given $\Phi = \{\varphi_i\}_{i=1}^m$, with $\varphi_i \in \mathbb{H}^n$, consider the frame potential. Then,

- Every local minimizer of the frame potential is also a global minimizer.
- If $m \leq n$, the minimum value of the frame potential is $FP(\{\varphi_i\}_{i=1}^m) = n$, and the minimizers are precisely the orthonormal sequences in $\mathbb{R}^n$.
- If $m \geq n$, the minimum value of the frame potential is $FP(\{\varphi_i\}_{i=1}^m) = m^2 n$, and the minimizers are precisely the unit norm tight frames for $\mathbb{R}^n$.

Therefore, the solution to the uniform-norm CDMA problem (minimizer of total interuser interference) are equal-norm tight frames.

III. Fundamental Tight Frame Inequality and Nonuniform-Norm CDMA [4]

Consider the following question: Given positive integers m and n, for what sequences of positive numbers $\{a_i\}_{i=1}^m$ do there exist tight frames $\{\varphi_i\}_{i=1}^m$ for $\mathbb{H}^n$, such that $\|\varphi_i\| = a_i$ for all i?

The following result provides a necessary condition upon the norms $\{a_i\}_{i=1}^m$.

Proposition 1: If $\{\varphi_i\}_{i=1}^m$ is a A-tight frame for $\mathbb{H}^n$ then,

$$\max_{i=1,\cdots,m} \|\varphi_i\|^2 \leq A = \frac{1}{n} \sum_{i=1}^{m} \|\varphi_i\|^2.$$

Proof: First note that for any $i = 1, \cdots, m$,

$$A\|\varphi_i\|^2 = \sum_{j=1}^{m} |\langle \varphi_i, \varphi_j \rangle|^2 \ \geq \|\varphi_i\|^4.$$

Thus, $\|\varphi_i\|^2 \leq A$ for all $i = 1, \cdots, m$, proving the inequality. For the equality, let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $\mathbb{H}^n$. By Parseval's identity,

$$\frac{1}{n} \sum_{i=1}^{m} \|\varphi_i\|^2 = \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n} |\langle \varphi_i, e_j \rangle|^2 = \frac{1}{n} \sum_{j=1}^{n} A\|e_j\|^2 = A.$$

■

Therefore, the norms $\{a_i\}_{i=1}^m$ of a tight frame of m elements for an n-dimensional space must satisfy the *fundamental tight frame inequality* (FTFI)

$$\max_{i=1,\cdots,m} a_i^2 \leq \frac{1}{n} \sum_{i=1}^{m} a_i^2. \tag{4}$$

Remarkably, this easily-found necessary condition turns out to be sufficient as well, although proof of sufficiency is much more difficult (see Thm. 2 below).

The following result, provides a lower bound for the frame potential (see Def. 1 above) similar to Thm. 1 but for the more general case of nonuniform-norm frame vectors. Note that this lower bound is not tight in general.

Proposition 2: For any $\mathbb{H}^n$ and any positive sequence $\{a_i\}_{i=1}^m$, the frame potential satisfies

$$\frac{1}{n} \left[\sum_{i=1}^{m} a_i^2 \right]^2 \leq FP(\{\varphi_i\}_{i=1}^m)$$

Furthermore, this lower bound is achieved if and only if $\{\varphi_i\}_{i=1}^m$ is tight for $\mathbb{H}^n$ with $\|\varphi_i\| = a_i$ for all i.

Proof: The result follows from Theorems 6.1 and 6.2 in [3]. ■

Note that the lower bound in Prop. 2 is only achieved when there exists a tight frame $\{\varphi_i\}_{i=1}^m$ for $\mathbb{H}^n$ with $\|\varphi_i\| = a_i$ for all i. From Prop. 1, we know that such frames cannot exist when the norms $\{a_i\}_{i=1}^m$ violate the FTFI in (4).

The following result by Cassaza et al. [4] resolves this ambiguity and determines the true minimum (and minimizer) of the frame potential even in cases where the FTFI is not satisfied.

Theorem 2: Let $\mathbb{H}^n$ be any n-dimensional Hilbert space. Given a positive decreasing sequence $\{a_i\}_{i=1}^m$ in $\mathbb{R}$, let i_0 denote the smallest index i for which

$$(n-i)a_i^2 \leq \sum_{j=i+1}^{m} a_j^2,$$

holds. Then, any local minimizer of the frame potential $FP(\{\varphi_i\}_{i=1}^m) : S(\{a_i\}_{i=1}^m) \to \mathbb{R}$ is of the form,

$$\{\varphi_1\} \perp \{\varphi_2\} \perp \cdots \perp \{\varphi_{i_0-1}\} \perp \{\varphi_i\}_{i=i_0}^m,$$

where $\{\varphi_i\}_{i=i_0}^m$ is a tight frame for its span.

Corollary 1: Let $\{a_i\}_{i=1}^m$ be a positive decreasing sequence. Then, there exists a tight frame $\{\varphi_i\}_{i=1}^m$ for $\mathbb{H}^n$ with $\|\varphi_i\| = a_i$ for all i if and only if $\{a_i\}_{i=1}^m$ satisfies the FTFI in (4).

It can also be shown that any local minimizer of the frame potential is a global minimizer and a closed-form expression for its optimal lower bound can be computed in terms of a_i's (see Cor. 2 in [4]).

Connection to CDMA. Consider a positive decreasing sequence $\{a_i\}_{i=1}^m$ with i_0 (defined in Thm. 2) at least one, i.e.,

$$a_1^2 > \frac{1}{n-1} \sum_{i=2}^{m} a_i^2.$$

Let $\{\varphi_i\}_{i=1}^m$ be a local minimizer of the frame potential. Thm. 2 guarantees that φ_1 is necessarily orthogonal to $\{\varphi_i\}_{i=2}^m$. Thus, as long as φ_1 is stronger than the "dimensional average" of the remaining (signature) vectors, then φ_1 is powerful enough to grab an entire dimension for itself, leaving the other points to fight over the remaining $n-1$ dimensions. The vectors $\{\varphi_i\}_{i=2}^m$ then repeat the above scenario and the whole sequence is characterized recursively.

In the general CDMA problem, the powers $\{p_i = a_i^2 = \|\varphi_i\|^2\}_{i=1}^m$ of the the individual users are predetermined (could vary with time) and are not necessarily equal. As described above, the problem of designing the optimal signature sequence $\{\varphi_i\}_{i=1}^m$ in minimum total-interference sense is equivalent to finding a frame with minimum frame potential subject to the power constraints. The results above indicate that, in the optimal solution, the "oversized" users (those with index $i < i_0$) are assigned orthogonal channels for their personal use. And for the remaining users, the designed signature vectors will form a tight frame.

If no user is oversized (i.e., no vector is much stronger than the rest), the problem reduces to finding a tight frame for $\mathbb{H}^n$ with norms $\{a_i = \sqrt{p_i}\}_{i=1}^m$. The results of this section characterize all solutions to this problem using a physical interpretation of frame theory.

Numerical Example. Fig. 2 gives a family of frames, each with its own set of three vector norms. The common feature among all is that they all achieve the lower bound for the frame potential and hence are optimal. The frame in Fig. 2(a) is the Mercedes-Benz (MB) unit-norm tight frame. The frames in panels (b) and (c) are also tight frames whereas the one in panel (d) is not. The arrangement in (d) has one vector which is too large (lacks balance compared to the MB frame) and its norms violate the FTFI. Therefore, although still optimal, it has lost the tightness property.

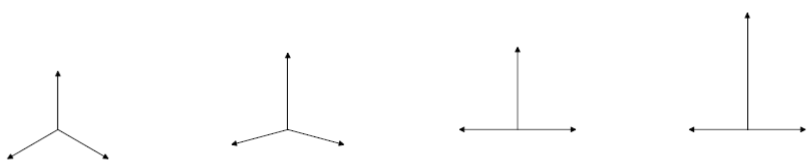


Fig. 2. (a) (b) (c) (d)

REFERENCES

[1] J. L. Massey and T. Mittelholzer, "Welch's bound and sequence sets for code-division multiple-access systems," *Sequences II: Methods in Communication, Security and Computer Sciences*, New York: Springer, 1993, pp. 63–78.

[2] J. Kovacevic and A. Chebira, "Life beyond bases: The advent of frames (Parts I and II)," *IEEE Signal Processing Mag.*, vol. 24, no. 4, pp. 86–104, 115–125, July 2007.

[3] J. J. Bendetto and M. Fickus, "Finite normalized tight frames." *Advances in Computational Mathematics*, vol. 18, pp. 357–385, Feb 2003.

[4] P. G. Cassaza and M. Fickus and K. Kovacevic and M. Leon and J. Tremain, "A physical interpretation for finite tight frames," *Harmonic Analysis and Applications*, Boston: Birkhauser, 2006, pp. 51–76.