A Brief Introduction to Grassmannian Frames

Arvind Ganesh

Notation

A set of N points in $\mathbb{R}^d$ is denoted by $X_d^N \doteq \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$, where each $\boldsymbol{x}_i \in \mathbb{R}^d$. The norm $\|\boldsymbol{x}\|$ denotes the usual Euclidean 2-norm. The inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. The transpose of a vector or matrix P is denoted by P^T, and its Hermitian transpose by P^*. The set of orthogonal bases for $\mathbb{R}^d$ is represented by SO_d, and the unit hypersphere by S^{d-1}.

1 Introduction

Given a *finite frame*[1] X_d^N, it is interesting to quantify the correlation between frame elements. In particular, it is desirable in some applications to construct frames with small correlation. The elements of SO_d represent the *ideal* scenario of zero correlation among frame elements.

Definition 1.1. The *maximum correlation* of a frame X_d^N, $\mathcal{M}_\infty(X_d^N)$, is defined as

$$\mathcal{M}_\infty(X_d^N) = \max_{k \neq l} \; |\langle \boldsymbol{x}_k, \boldsymbol{x}_l \rangle|$$

Evidently, $\mathcal{M}_\infty(X_d^N) \in [0, 1]$ for a normalized frame X_d^N. $\mathcal{M}_\infty(X_d^N) = 0$ iff X_d^N is an orthonormal basis for $\mathbb{R}^d$.

Definition 1.2. A unit norm finite frame for $\mathbb{R}^d$, $U_d^N = \{\boldsymbol{u}_k\}_{k=1}^N$, is an (N, d)-*Grassmannian frame* if

$$\mathcal{M}_\infty(U_d^N) = \inf\{\mathcal{M}_\infty(X_d^N)\}, \qquad (1)$$

where the infimum is taken over all unit norm, N-element frames for $\mathbb{R}^d$.

Thus, Grassmannian frames are frames satisfying a min-max correlation criterion, as given by (1). It can be proved that Grassmannian frames exist for any given d and N ($N \geq d$). A brief sketch of the proof is given below.

Proof. (Existence of Grassmannian frames)
Define the function

$$f \;:\; \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{N \text{ times}} \to \mathbb{R}^+$$

$$f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = \mathcal{M}_\infty(\{\boldsymbol{x}_k\}_{k=1}^N)$$

Now, f is a continuous function on $X := \mathbb{R}^d \times \ldots \times \mathbb{R}^d$, and the set of normalized N-point frames in $\mathbb{R}^d$ is a compact set in X. Hence, f is continuous on a compact set $S^{d-1} \times \ldots \times S^{d-1}$ (N times), and so f achieves its absolute minimum on this set. Thus, Grassmannian frames exist for any $N \geq d$. □

Though it is clear that Grassmannian frames exist, constructing one is a challenging problem.

[1] Span of X_d^N is $\mathbb{R}^d$, which necessitates $N \geq d$.

2 Grassmannian frames in $\mathbb{R}^2$

Theorem 2.1. **($(N, 2)$-Grassmannian).** *Let $X = X_2^N$ be a unit norm frame in $\mathbb{R}^2$. Then, we have the lower bound*

$$\mathcal{M}_\infty(X) \geq \cos\left(\frac{\pi}{N}\right) \qquad (2)$$

Furthermore, X is an $(N, 2)$-Grassmannian frame iff there is a $P \in SO_2$ and a sequence $\{\epsilon_k\}_{k=1}^N \subset \{\pm 1\}^N$ such that

$$P(\epsilon X) := \{P(\epsilon_k \boldsymbol{x}_k) \;:\; \boldsymbol{x}_k \in X_2^N\} \qquad (3)$$

$$= \left\{ \begin{pmatrix} \cos(\pi k/N) \\ \sin(\pi k/N) \end{pmatrix} \right\}_{k=1}^N \qquad (4)$$

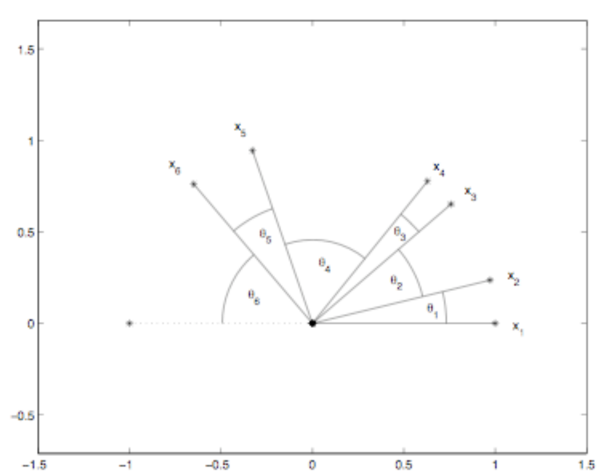

Figure 1: Reordering of points

Proof. Since $\mathcal{M}_\infty(X)$ is not affected by rotations and by the sign of $\boldsymbol{x}_k \in X$, we assume without loss of generality that all points lie in the upper half plane, with $\boldsymbol{x}_1$ aligned horizontally. The points are then reordered in clockwise direction, as shown in Figure 1. For $k = 1, \dots, N$-1, let θ_k be the angle between $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$, and let θ_N be the angle between $\boldsymbol{x}_N$ and the negative x-axis. This reordering gives us the relation $\sum_{k=1}^{N} \theta_k = \pi$.

$$\begin{aligned}\mathcal{M}_\infty(X) &= \max_{k\neq l} |\langle \boldsymbol{x}_k, \boldsymbol{x}_l\rangle| \\ &= \max_{k\neq l} \left|\cos\left(\sum_{j=l}^{k-1}\theta_j\right)\right| \\ &= \left|\cos\left(\min_{k=1,\dots,N}\theta_k\right)\right|\end{aligned}$$

Thus, in order to minimize $\mathcal{M}_\infty(X)$, we would have to maximize $\min_{k=1,\dots,N}\theta_k$. Assume that the θ_k's that minimize $\mathcal{M}_\infty(X)$ are not all equal. Then, $\exists\ m \in \{1, \dots, N\text{-}1\}$ so that

$$\theta_{k_1} = \dots = \theta_{k_m} < \theta_{k_{m+1}} \le \dots \le \theta_{k_N}$$

Let $\nu = \theta_{k_{m+1}} - \theta_{k_m}$. Define the sequence β_k as

$$\beta_{k_j} = \begin{cases} \theta_{k_j} + \frac{\nu}{2m} & \text{for } j = 1, \dots, m, \\ \theta_{k_j} - \frac{\nu}{2} & \text{for } j = m+1, \\ \theta_{k_j} & \text{for } j = m+2, \dots, N \end{cases}$$

Now the new set has a strictly larger minimum angle than the original, i.e., it yields a smaller $\mathcal{M}_\infty(X)$, which is a contradiction. Therefore, the original θ_k's are all equal. Since they sum to π, we have $\theta_1 = \dots = \theta_N = \pi/N$. Thus, $\mathcal{M}_\infty(X) \ge \cos(\pi/N)$. The minimizer gives us the $(N, 2)$-Grassmannian frame. The proof of the second part of the theorem follows from the fact that the $(N, 2)$-Grassmannian frame is the set of N^{th} roots of unity, upto a rotation and sign change.[2] □

3 A Lower Bound for $\mathcal{M}_\infty$

Theorem 3.1. *Let $N \ge d$, $X_d^N \subset S^{d-1}$, and let $d_0 = dim(span(X_d^N))$. Then*

$$\mathcal{M}_\infty(X_d^N) \ge \sqrt{\frac{N - d_0}{d_0(N-1)}}, \tag{5}$$

where equality holds iff X_d^N is equiangular, and X_d^N is a tight frame for its span with frame bounds $A = B = \frac{N}{d_0}$. Furthermore, if $N > \frac{d(d+1)}{2}$, then X_d^N is not equiangular, hence equality cannot hold in (5).

[2] For N even, the N^{th} roots of unity actually form a (N/2,2)-Grassmannian frame.

Proof. Let $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N$ be the eigenvalues of the Grammian matrix G. Hence,

$$\sum_{k=1}^{d_0} \lambda_k = \text{Tr}(G) = \sum_{k=1}^{N} |\langle \boldsymbol{x}_k, \boldsymbol{x}_k\rangle| = N$$

Now set $e_k = \lambda_k - \frac{N}{d_0}$, $k = 1, \dots, d_0$. Then,

$$\sum_{k=1}^{d_0} \lambda_k^2 = \frac{N^2}{d_0} + \sum_{k=1}^{d_0} e_k^2 \ge \frac{N^2}{d_0}$$

with equality iff $\lambda_1 = \dots = \lambda_{d_0} = \frac{N}{d_0}$.

$$\sum_{k=1}^{N}\sum_{l=1}^{N} |\langle \boldsymbol{x}_k, \boldsymbol{x}_l\rangle|^2 = \text{Tr}(G^2) = \sum_{k=1}^{d_0} \lambda_k^2$$

Since $|\langle \boldsymbol{x}_k, \boldsymbol{x}_l\rangle| = |\langle \boldsymbol{x}_l, \boldsymbol{x}_k\rangle|$, we have

$$\begin{aligned}\frac{N^2}{d_0} &\le \sum_{k=1}^{N}\sum_{l=1}^{N} |\langle \boldsymbol{x}_k, \boldsymbol{x}_l\rangle|^2 \\ &= N + 2\sum_{k<l} |\langle \boldsymbol{x}_k, \boldsymbol{x}_l\rangle|^2 \\ &\le N + 2\,\frac{N(N-1)}{2}\,\max_{k\neq l}\{|\langle \boldsymbol{x}_k, \boldsymbol{x}_l\rangle|^2\}\end{aligned}$$

Rearranging the terms, we get (5). □

Definition 3.2. Let $d \le N \le \frac{d(d+1)}{2}$. Let X_d^N be a unit norm frame for $\mathbb{R}^d$. We call X_d^N an *optimal Grassmannian frame* if X_d^N satisfies (5) with equality, i.e.,

$$\mathcal{M}_\infty(X_d^N) = \sqrt{\frac{N - d_0}{d_0(N-1)}}$$

Optimal Grassmannian frames may not exist for some pairs (N, d). For example, (5,3)-Grassmannian frames are not optimal, while (3,3), (4,3) and (6,3)-Grassmannians are optimal.

References

[1] J. J. Benedetto and J. D. Kolesar, *Geometric properties of Grassmannian frames for $\mathbb{R}^2$ and $\mathbb{R}^3$*, EURASIP J. Applied Signal Processing, 2006.

[2] T. Strohmer and R. W. Heath, Jr., *Grassmannian frames with applications to coding and communication*, Applied and Computational Harmonic Analysis, vol. 14, no. 3, 2003.