Introduction to “Finite normalized tight frames”

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A finite normalized tight frame (FNTF) in a real or complex Euclidean space K^d ($K = \mathbb{R}$ or $\mathbb{C}$) is a Bessel sequence $\{x_n\}_{n=1}^N$ that generalizes orthonormal bases that maintains the decomposition property

$$y = \frac{1}{A} \sum_{n=1}^{N} \langle y, x_n \rangle x_n \;\forall y \in K^d \qquad (1)$$

while giving up the basis property.

1 Properties of FNTFs

A FNTF is a Bessel sequence $\{x_n\}_{n=1}^N$ in a finite dimensional Hilbert space H of dimension d that is

- A tight frame (Theorem 2.1):

(a) $\{x_n\}_{n=1}^N$ is a A-tight frame with frame operator $S \iff S = AI$ where A is a positive constant and I is the identity map. (I.e. Eq. 1 holds.)

(b) $\{x_n\}_{n=1}^N$ is a normalized A-tight frame if $A \geq 1$, with $A = 1$ if and only if $\{x_n\}_{n=1}^N$ is an orthonormal basis for H.

- Normalized: each element has $\|x_n\| = 1$.
- Finite (Theorem 2.3): A normalized Bessel sequence in a finite dimensional Hilbert space H must be a sequence of finite length.

A FNTF is a generalization of an orthonormal basis in the sense that the frame operator $S = L^* L$ is proportional to the identity and the Grammian operator $G = LL^*$, $G(m,n) = \langle x_m, x_n\rangle$ has diagonal entries equal to one.

The frame constant A is a measure of the degeneracy of the FNTF.

Theorem 3.1 (value of frame constant). If $\{x_n\}_{n=1}^N$ is a FNTF for a Hilbert space of dimension d, then it has frame constant $A = \frac{N}{d}$.

Theorem 3.2 (existence). Given any finite-dimensional space K^d, there always exists a FNTF with $N \geq d$ elements.

2 Examples

Roots of unity: The N^{th} roots of unity form a FNTF for $\mathbb{R}^2$, e.g. the (normalized) Mercedes frame from class.

Regular solids: Vectors in $\mathbb{R}^3$ pointing to the vertices of regular solids inscribed within a unit sphere S^2 form a FNTF for $\mathbb{R}^3$. Examples of regular solids are the tetrahedron, cube, octahedron, dodecahedron, icosahedron and soccer ball.

Examples in $\mathbb{R}^3$ generalize the geometric regularity of the roots of unity to 3 dimensions.

3 Spherical equidistribution and the frame potential

How do we construct FNTFs? They turn out to be solutions (but not the *only* solutions) to the following problem.

Problem (spherical equidistribution). Place N points on a sphere $S^{d-1} \subset \mathbb{R}^d$ such that the points are "as far away as possible".

In $\mathbb{R}^2$, the solutions are the N^{th} roots of unity. In $\mathbb{R}^{\geq 3}$, there is no unambiguous way to define "as far away as possible".

Definition (frame force). The frame force (FF) is a vector function

$$FF \quad : \quad S^{d-1} \times S^{d-1} \to \mathbb{R}^d \tag{2}$$

$$FF(a,b) = \langle a,b \rangle (a-b) \tag{3}$$

which is a central force with $f(x) = 1 - \frac{1}{2}x^2$.

Since the frame force is a central force, it can be expressed as the negative gradient of a scalar potential function.

Definition (frame potential). The frame potential (FP) is a scalar function defined for a finite Bessel sequence

$$FF \quad : \quad S\left(K^d\right) \to [0,\infty) \tag{4}$$

$$FP\left(\{x_n\}_{n=1}^N\right) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_n, x_m \rangle|^2 \tag{5}$$

which corresponds to the potential $p(x) = -\frac{x^2}{2}\left(1 - \frac{x^2}{4}\right)$ which generates a force $f(x) = -\frac{p(x)}{x}$.

Theorem 6.1. (relation between frame potential and frame operator) For a finite Bessel sequence $\{x_n\}_{n=1}^N$ with frame operator S, the frame potential is

$$FP\left(\{x_n\}_{n=1}^N\right) = \operatorname{tr} S^2 \tag{6}$$

4 The main result: FNTFs minimize the frame potential

Theorem 6.2. (bounds on frame potential) Given any positive integers d and N and a normalized Bessel sequence $\{x_n\}_{n=1}^N \subseteq S\left(K^d\right)$, then

$$N \max\left(1, \frac{N}{d}\right) \leq FP\left(\{x_n\}_{n=1}^N\right) \leq N^2 \tag{7}$$

The lower bound is attained if and only if $\{x_n\}_{n=1}^N$ is an orthonormal set (if $N \leq d$, with lower bound N) or $\{x_n\}_{n=1}^N$ is a FNTF (if $N \geq d$, with lower bound $\frac{N^2}{d}$) for K^d.

Proof. The upper bound follows from the Cauchy-Schwarz inequality

$$FP\left(\{x_n\}_{n=1}^N\right) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_n, x_m \rangle|^2 \leq \sum_{m=1}^N \sum_{n=1}^N \|x_m\|^2 \|x_n\|^2 = N^2 \tag{8}$$

The first lower bound ($\geq N$) follows from the non-negativity of $|\langle x_n, x_m \rangle|$ and the normalization since

$$FP\left(\{x_n\}_{n=1}^N\right) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_n, x_m \rangle|^2 \geq \sum_{n=1}^N |\langle x_n, x_n \rangle|^2 = N \tag{9}$$

The second lower bound ($\geq N^2/d$) requires the frame operator S, which has non-negative eigenvalues $\{\lambda_k\}_{k=1}^d$. Then Theorem 6.1 and the definition of trace gives

$$FP\left(\{x_n\}_{n=1}^N\right) = \operatorname{tr} S^2 = \sum_{k=1}^d \lambda_k^2 \tag{10}$$

We want to minimize this subject to the normalization constraint

$$\sum_{k=1}^{d} \lambda_k = \text{tr}\, S = N \tag{11}$$

The unique minimizer occurs when all λ_k are equal, i.e. $\lambda_k = \frac{N}{d}$ for all k. Then $S = \frac{N}{d} I$ and by Theorem 2.1, $\{x_n\}_{n=1}^{N}$ is a FNTF with corresponding frame potential $FP\left(\{x_n\}_{n=1}^{N}\right) = \sum_{k=1}^{d} \left(\frac{N}{d}\right)^2 = \frac{N^2}{d}$.

Minimizing the frame potential gives us solutions which are either FNTFs or orthonormal sequences.

Theorem 7.4 (local minimizers of the frame potential) A local minimizer for the frame potential is either an orthonormal set or a FNTF for K^d.

Theorem 7.1 (global minimizers of the frame potential) Given any positive integers d and N and a frame potential defined in Eq. 4-5, then

(a) Every local minimizer of the frame potential is also a global minimizer.

(b) If $N \le d$, then the minimum value of the frame potential is N, and the minimizing sequences are orthonormal sets in K^d.

(c) If $N \ge d$, then the minimum value of the frame potential is $\frac{N^2}{d}$, and the minimizing sequences are FNTFs in K^d.

Proof. We first prove (b), then (c), then (a). First, the frame potential is a continuous function on a compact set and therefore has a global minimizer, which is also a local minimizer. Theorem 7.4 then states that this is either an orthonormal set or a FNTF. For $N \le d$ the local minimizers must be orthonormal sets: if $N < d$ they cannot be FNTFs, and the case $N = d$ proceeds from Theorem 2.1. Second, for $N \ge d$ the local minimizers must be FNTFs: if $N > d$ they cannot be orthonormal, and the case $N = d$ proceeds from Theorem 2.1. Third, these local minimizers are also global minimizers because they attain the lower bound given by Theorem 6.2.

References

[1] J. J. Benedetto and M. Fickus, "Finite normalized tight frames", *Adv. Comput. Math.* **18**: 357-385, 2003.