1 Introduction

When working in infinite dimensions, finding the inverse of the frame operator is extremely difficult. Inverting finite-dimensional operators is in general a much more manageable operation. It would be helpful, then, if we could approximate the inverse frame operator by a sequence of finite-dimensional operators.

Throughout this talk, we will always consider our frames to be a frame $\{f_i\}_{i \in I}$ together with an increasing sequence of finite subsets $\{I_n\} \nearrow I$.

2 The Projection Method

Given $\{f_i\}_{i \in I}$ a frame for H, we'll write $H_n = \text{span}\{f_i : i \in I_n\}$ and let P_n denote the projection of H onto H_n. As I_n is finite for any n, $\{f_i\}_{i \in I_n}$ is a frame for H_n. We'll write S_n for the corresponding frame operator. Then we see that

$$P_n f = \sum_{i \in I} \left\langle P_n f, S_n^{-1} f_i \right\rangle f_i = \sum_{i \in I_n} \left\langle f, S_n^{-1} f_i \right\rangle f_i.$$

The $P_n f$ approximate f, so we hope that $\langle f, S_n^{-1} f_i \rangle$ will approximate the frame coefficients of f. That is,

$$\left\langle f, S_n^{-1} f_i \right\rangle \to \left\langle f, S^{-1} f_i \right\rangle \qquad \text{as } n \to \infty \text{ for all } i \in I.$$

If this is the case, we say the projection method works for $\{f_i\}_{i \in I}$.

Theorem 1. *Given a frame $\{f_i\}_{i \in I}$, the projection method works if and only if for every $j \in I$, there is a constant c_j such that $\|S_n^{-1} f_j\| \leq c_j$ for every n with $j \in I_n$.*

Proof. Largely computational. See [1]. □

Examples.

1. The projection method works for all Riesz bases.
2. An example when the projection method does not work: Start with an ONB $\{e_j\}$, and set $f_1 = e_1$, $f_i = e_{i-1} + (1/i)e_i$ for $i \geq 2$. It can be shown (see [1]) that $\{f_i\}_{i \in N}$ is a frame with bounds $1 - \sqrt{\pi^2/6 - 1}$ and 3, but also that $\|S_n^{-1} f_1\| \to \infty$ as $n \to \infty$.

3 The Strong Projection Method

Note that the rates of convergence of the frame coefficients in Theorem 1 are not necessarily the same. One way to improve this situation is to instead require that

$$\{\left\langle f, S_n^{-1} f_i \right\rangle\}_{i \in I_n} \to \{\left\langle f, S^{-1} f_i \right\rangle\}_{i \in I} \qquad \text{in } l_2(I).$$

If this is the case, we will say that the strong projection method works.

In Theorem 1, our bounds for $\|S_n^{-1} f_j\|$ depended on j; if for a given frame the bounds are in fact independent of j, we expect we can show that the strong projection works. To this end, we consider the following

Definition. Given a frame, we call $\{f_i\}_{i\in I}$ a *conditional Riesz frame* if every $\{f_i\}_{i\in I_n}$ has a common lower frame bound, or equivalently if $\sup_n \|S_n^{-1}\| < \infty$.

In [2], Casazza and Christensen showed several equivalent conditions for the strong projection method working.

Theorem 2. *Let $\{f_i\}_{i\in I}$ be a frame and $\{I_n\} \nearrow I$ an increasing sequence of finite sets. Then the following are equivalent.*

1. *The strong projection method works.*
2. *$\{f_i\}_{i\in I}$ is a conditional Riesz frame.*
3. *$S_n^{-1} P_n f \to S^{-1} f$ for all $f \in H$. (Convergence in the strong operator topology.)*
4. *$\langle S_n^{-1} P_n f, g\rangle \to \langle S^{-1} f, g\rangle$ for all $f, g \in H$. (Convergence in the weak operator topology.)*

Proof. We will show only part of the proof. "(2) ⇒ (3) ⇒ (1)" Suppose $\|S_n^{-1}\| \le C$.

$$\begin{aligned}
&\|\{\langle f, S_n^{-1} f_i\rangle\}_{i\in I_n} - \{\langle f, S^{-1} f_i\rangle\}_{i\in I}\|^2_{l_2(I)} \\
&\qquad = \sum_{i\in I_n} |\langle f, S_n^{-1} f_i\rangle - \langle f, S^{-1} f_i\rangle|^2 + \sum_{i\notin I_n} |\langle f, S^{-1} f_i\rangle|^2
\end{aligned}$$

The second term vanishes as $n \to \infty$. Considering the first term as $n \to \infty$, we see

$$\begin{aligned}
\sum_{i\in I_n} |\langle f, S_n^{-1} f_i\rangle - \langle f, S^{-1} f_i\rangle|^2 &= \sum_{i\in I_n} |\langle f, P_n S_n^{-1} f_i\rangle - \langle f, S^{-1} f_i\rangle|^2 \\
&= \sum_{i\in I_n} |\langle S_n^{-1} P_n f - S^{-1} f, f_i\rangle|^2 \qquad \text{(by self-adjointness)} \\
&\le B\|S_n^{-1} P_n f - S^{-1} f\|^2 \qquad \text{(where B is the upper frame bound)} \\
&\le B\left(\|S^{-1} f - P_n S^{-1} f\| + \|P_n S^{-1} f - S_n^{-1} P_n f\|\right)^2 \\
&= B\left(\|S^{-1} f - P_n S^{-1} f\| + \left\|\sum_{i\in I_n} \langle S^{-1} f, f_i\rangle S_n^{-1} f_i - S_n^{-1} P_n f\right\|\right)^2 \\
&\qquad \text{(using the dual frame expansion)} \\
&\le B\left(\|S^{-1} f - P_n S^{-1} f\| + \|S_n^{-1}\| \left\|\sum_{i\in I_n} \langle S^{-1} f, f_i\rangle f_i - P_n f\right\|\right)^2
\end{aligned}$$

$$\leq B\left(\|S^{-1}f - P_nS^{-1}f\| + C\left\|\sum_{i\in I_n}\left\langle S^{-1}f, f_i\right\rangle f_i - P_nf\right\|\right)^2$$

We note $P_nS^{-1}f \to S^{-1}f$ and $P_nf \to f$ as $n \to \infty$. Because $\{f_i\}_{i\in I}$ is a frame, we have $\sum_{i\in I_n}\langle S^{-1}f, f_i\rangle f_i \to f$ as $n \to \infty$. Hence this last line vanishes as $n \to \infty$. Thus we have convergence in the strong operator topology, and the strong projection method works.

Other directions follow from applications of Cauchy-Schwarz and the Uniform Boundedness principle; see [4]. □

Remark (A method that works for any frame and sequence of indexing sets. [3]). Given our frame and the sequence of I_n, we choose an appropriate $J_n \supseteq I_n$. Instead of S_n^{-1}, we consider $\widetilde{S_n}^{-1}$, where $\widetilde{S_n}$ is the frame operator for the frame $\{P_nf_j\}_{j\in J_n}$. Then $\widetilde{S_n}^{-1}P_nf \to S^{-1}f$ for all $f \in H$.

Examples.

1. The strong projection method works for all Riesz bases.

2. The ordering of the frame elements matters. Let $\{e_i\}$ be an ONB for H. We first consider the frame $\{f_i\}_{i\in I}$ given by

$$f_{2i-1} = e_i \qquad f_{2i} = \frac{1}{i}e_i.$$

together with $I_n = 1, 2, ..., n$. Then we can estimate the lower frame bound for $\{f_i\}_{i\in I_n}$ as follows. Take $f \in H_n$; note $e_1, \dots, e_{\lceil n/2\rceil}$ is a basis for this subspace. Then

$$\begin{aligned}\sum_{i=1}^{n}|\langle f, f_i\rangle|^2 &= \sum_{i=1}^{\lceil n/2\rceil}|\langle f, e_i\rangle|^2 + \sum_{i=1}^{\lfloor n/2\rfloor}\left|\left\langle f, \frac{1}{i}e_i\right\rangle\right|^2 \\ &= \sum_{i=1}^{\lfloor n/2\rfloor}\left(1+\frac{1}{i^2}\right)|\langle f, e_i\rangle|^2 \quad (+ \quad \text{possibly} \quad |\langle f, e_{\lceil n/2\rceil}\rangle|^2) \\ &\geq \sum_{i=1}^{\lceil n/2\rceil}|\langle f, e_i\rangle|^2 = \|f\|_{H_n}^2,\end{aligned}$$

giving us a uniform lower frame bound. Thus this frame is a conditional Riesz frame and the strong projection method works.

On the other hand, if we take the same frame elements together with $I_n = \{2, 1, \dots, 2k, 2k-1, \dots, n\}$ (or equivalently, swapping the definitions of f_{2k} and f_{2k-1}) we observe that

$$\sum_{i=1}^{2k}|\langle e_k, f_i\rangle|^2 = \left|\left\langle e_k, \frac{1}{k}e_k\right\rangle\right|^2 = \frac{1}{k^2} \to 0 \qquad \text{as } k \to \infty,$$

and hence this frame is not a conditional Riesz frame.

References

[1] Ole Christensen, *Frames and the Projection Method*, Appl. Comp. Harm. Anal. **1**, (1993) p. 50-53

[2] Peter G. Casazza, Ole Christensen, *Riesz Frames and Approximation of the Frame Coefficients*, Approx. Theory & its Appls., **14:2** (June 1998) 1-11

[3] Ole Christensen, *Finite-dimensional approximation of the inverse frame operator*, J. Four. Anal. Appl. 6, **1** (2000), 79-91

[4] Peter G. Casazza, Ole Christensen, *Approximation of the frame coefficients using finite-dimensional methods*, J. Elec. Imaging, Vol 6 (1997), 479-483, 1997