Application of Frames to Erasure Channels

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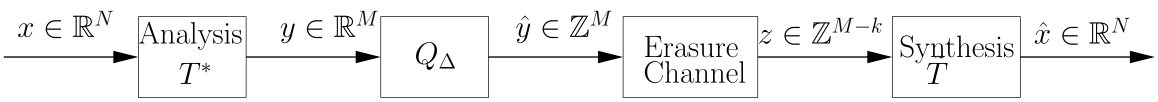
1 Quantized Erasure Channels



$\{f_i\}_{i=1}^M$ is a frame for $\mathbb{R}^N$, *i.e.*, $\text{span}\{f_i\}_{i=1}^M = \mathbb{R}^N$ $(M \geqslant N)$

- Analysis: $T^*x = \{\langle x, f_i\rangle\}_{i=1}^M$. Define y
- Quantization: $\hat{y}_i = Q_\Delta y_i = \lceil \frac{y_i}{\Delta} \rfloor \Delta$ (round-off to the nearest integer multiple of Δ)
- Erasure channel: Upto k quantized frame coefficients are lost (erased) during transmission. Define the erasure set $J \subset \{1, 2, \dots, M\}$ to be the index set for the erased coefficients; $|J| \leqslant k$.
- Reconstruction: $\hat{x} = \tilde{T}\{\hat{y}_i\}_{i \in J^c} = \sum_{i \in J^c} \hat{y}_i g_i, \quad g_i \in \mathbb{R}^N$

1.1 Robustness to erasures

In the absence of quantization, in order for the scheme to be robust to erasure of the coefficients $\{y_i\}_{i \in J}$ for a particular erasure set J, we need $\{f_i\}_{i \in J^c}$ to be a frame for $\mathbb{R}^N$, which is equivalent to requiring $\text{span}\{f_i\}_{i \in J^c} = \mathbb{R}^N$. Then $\{g_i\}_{i \in J^c}$ is a dual frame for $\{f_i\}_{i \in J^c}$.

A frame $\{f_i\}_{i=1}^M$ is said to be robust to k-erasures if the communication is robust to deletion of the coefficients $\{f_i\}_{i \in J}$ for *any* J that has k or fewer elements.

1.2 Effect of quantization

By Cauchy-Schwartz, for any $i = 1, 2, \dots, M$, $|y_i|^2 \leqslant \|x\|_2^2 \cdot \|f_i\|_2^2$. Therefore, if $\|x\|_2^2 \leqslant E$, then, $|\hat{y}_i| \in \{0, \Delta, 2\Delta, \dots, K_i\Delta)\}$ where $K_i \approx \sqrt{E}\,\|f_i\|_2/\Delta$. In order for the "alphabet-size" for all $\hat{y}_i$'s to be equal, we require the frame $\{f_i\}_{i=1}^M$ to be *equal-norm*, *i.e.*, $\|f_i\|_2 = \|f_j\|_2$, $\forall i, j \in \{1, 2, \cdots M\}$.

We have, $\hat{y}_i = y_i + \nu_i$; with $\nu_i \in (-0.5, 0.5]$. As is typical, we model ν_i as independent random variables distributed uniformly over their range and hence they have zero mean and variance $\Delta^2/12$. Now, in the absence of erasures,

$$\begin{aligned}
\hat{x} &= \sum_{i=1}^{M} \hat{y}_i g_i = \underbrace{\sum_{i=1}^{M} \langle x, f_i \rangle g_i}_{x} + \sum_{i=1}^{M} \nu_i g_i \\
\Rightarrow \quad \|x - \hat{x}\|_2^2 &= \sum_{i=1}^{M} \sum_{j=1}^{M} \nu_i \nu_j \langle g_i, g_j^* \rangle \\
\Rightarrow \quad \mathrm{E}\|x - \hat{x}\|_2^2 &= \frac{\Delta^2}{12} \sum_{i=1}^{M} \|g_i\|_2^2 \geqslant \frac{\Delta^2}{12} \mathrm{tr}\left(S^{-1}\right) = \frac{\Delta^2}{12} \sum_{i=1}^{N} \frac{1}{\lambda_i}
\end{aligned} \tag{1}$$

where λ_i are the eigenvalues of S, the frame operator for the frame $\{f_i\}_{i=1}^M$. (Note: the inequality in the above relation is an equality if and only if $\{g_i\}_{i=1}^M$ is the canonical dual frame of $\{f_i\}_{i=1}^M$).

Now, for an equal-norm frame with M frame vectors of norm $\sqrt{r}$, the sum of the eigenvalues of the frame operator S is Mr. Thus minimizing the MSE is equivalent to minimizing $\sum_{i=1}^N 1/\lambda_i$ under the constraint $\sum_{i=1}^N \lambda_i = Mr$. The minima is attained when all the eigenvalues are equal to $(M/N)r$, which is true if and only if the frame is *tight* with frame bounds $A = B = (M/N)r$ (Theorem 3.1 in [1]).

1.3 Summary

For reliable communication over a quantized erasure channel, we desire an equal-norm, tight frame that is robust to k-erasures. Furthermore, note that for any tight frame with frame bounds $A = B$, we can find an equivalent *Parseval tight frame* with with frame bounds $A = B = 1$, by scaling the frame vectors by $\sqrt{A}$. In that case, we can find an orthonormal basis $\{e_i\}_{i=1}^M$ for $\mathbb{R}^M$ such that :

$$\begin{bmatrix} f_i \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} I_{N\times N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{P} e_i, \quad i = 1, 2, \ldots, M \tag{2}$$

Note that P is a rank-N projection operator (see Theorem 1.3.2 in [2]).

2 Conditions for the robustness to k-erasure

Lemma 1. *[3] Let $\{e_i\}_{i=1}^M$ be an orthonormal basis for $\mathcal{H}_M$. Let P be an orthogonal projection of $\mathcal{H}_M$ onto N-dimensional subspace $\mathcal{H}_N$. Fix $J \subset \{1, 2, \cdots, M\}$ with $|J| = k \leqslant M - N$ and let $\mathcal{K} = \mathrm{span}\{e_i\}_{i \in J^C}$. For an orthonormal basis $\{\varphi_j\}_{j=1}^{M-N}$ for $\mathcal{H}_N^\perp$, we consider the matrix $A \in \mathbb{C}^{(M-N)\times M}$ defined by*

$$[A]_{i,j} \triangleq \langle \varphi_i, e_j \rangle, \ i = 1, \cdots, M - N, j = 1, \cdots, M. \tag{3}$$

Then the followings are equivalent:

- *(a) $\{Pe_i\}_{i=1}^M$ is robust to the erasure of the element $\{Pe_i\}_{i \in J}$.*
- *(b) $\mathrm{rank}([A]_{\bullet,J}) = k$, where $[A]_{\bullet,J}$ denotes the minor with the columns of A indexed by J.*

Proof. First we note that $\mathcal{K} = P\mathcal{K} \oplus (I - P)\mathcal{K}$,

$$\begin{aligned}
M - k &= \dim \mathcal{K} = \dim P\mathcal{K} + \dim(I - P)\mathcal{K} \\
\Rightarrow \quad \dim(I - P)\mathcal{K} &= M - \dim P\mathcal{K} - k.
\end{aligned} \tag{4}$$

Also since $\mathcal{H}_N = P\mathcal{H}_M$,

$$\begin{aligned}
\mathcal{H}_N^\perp &= (I - P)\mathcal{H}_M = (I - P)(\mathcal{K} \oplus \mathcal{K}^\perp) = (I - P)\mathcal{K} + (I - P)\mathcal{K}^\perp \\
\Rightarrow \quad M - N &= \dim \mathcal{H}_N^\perp \leqslant \dim(I - P)\mathcal{K} + \dim(I - P)\mathcal{K}^\perp.
\end{aligned} \tag{5}$$

From (4) and (5), we see that

$$\dim P\mathcal{K} - N + k \leqslant \dim(I-P)\mathcal{K}^{\perp}. \tag{6}$$

Now (a) holds $\iff P\mathcal{K} = P\mathcal{H}_M \iff \dim P\mathcal{K} = N$, which from (6) is equivalent to

$$\dim(I-P)\mathcal{K}^{\perp} = k, \tag{7}$$

where we have used the fact that $\dim(I-P)\mathcal{K}^{\perp} \leqslant \dim \mathcal{K}^{\perp} = k$.

Define $\Phi^* : \mathcal{H}_M \to \mathbb{C}^{M-N}$ by

$$(\Phi^* f)_i = \langle f, \varphi_i \rangle, \forall i = 1, 2, \cdots, M-N, \tag{8}$$

and $\Phi : \mathbb{C}^{M-N} \to \mathcal{H}_M$ by

$$\Phi c = \sum_{i=1}^{M-N} c_i \varphi_i. \tag{9}$$

Then $(I-P) = \Phi\Phi^*$, and $\dim(I-P)\mathcal{K} = \dim \Phi\Phi^*\mathcal{K}^{\perp}$.

Also, since $\ker(\Phi) = \{0\}$, there exists Φ^L such that $\Phi^L\Phi = I$. Hence

$$\dim \Phi^L\Phi\Phi^*\mathcal{K}^{\perp} = \dim \Phi^*\mathcal{K}^{\perp} \leqslant \dim \Phi\Phi^*\mathcal{K}^{\perp} \leqslant \dim \Phi^*\mathcal{K}^{\perp}. \tag{10}$$

and $\dim(I-P)\mathcal{K} = \dim \Phi^*\mathcal{K}^{\perp}$.

Therefore, $\dim \Phi^*\mathcal{K}^{\perp} = k$ is equivalent to (a). We note that $\Phi^*\mathcal{K}^{\perp} = [A]_{\bullet,J}\mathbb{C}^k$, and therefore (b) is equivalent to (a). □

Theorem 1. *With the notation of Lemma 1, the followings are equivalent:*

(a) $\{Pe_i\}_{i=1}^M$ *is robust to k-erasure.*

(b) For every $J \subset \{1, 2, \cdots, M\}$ *with* $|J| = k$, $\operatorname{rank}([A]_{\bullet,J}) = k$.

Proof. The results directly follows from Lemma 1. □

Theorem 2. *With the notation of Lemma 1, the followings are equivalent:*

(a) $\{Pe_i\}_{i=1}^M$ *is equal-norm (and hence* $\{(I-P)e_i\}_{i=1}^M$ *is equal-norm).*

(b) For every $1 \leqslant i \leqslant M$ *we have,*

$$[A^*A]_{i,i} = \sum_{j=1}^{M-N} |\langle \varphi_j, e_i \rangle|^2 = \frac{M-N}{M}. \tag{11}$$

Example (Harmonic or Fourier Frame)
Consider $\mathcal{H}_M = \mathbb{C}^M$, and let F_M be the $M \times M$ DFT matrix defined by

$$[F_M]_{k,\ell} = \frac{1}{\sqrt{M}} e^{j2\pi k\ell/M}. \tag{12}$$

Note that the columns of F_M ($\{e_i\}_{i=1}^M$) form an orthonormal basis for $\mathbb{C}^M$ and $F_M^{-1} = F_M^H$.

Now, consider $\mathcal{H}_N \subset \mathbb{C}^M$, such that

$$f = [f(1), f(2), \ldots, f(M)]^T \in \mathcal{H}_N \iff f(k) = 0, \forall k = N+1, N+2, \ldots, M. \tag{13}$$

Then the projection from $\mathbb{C}^M$ onto $\mathcal{H}_N$ is

$$P = \begin{bmatrix} I_{N,N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{14}$$

and

$$I - P = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{M-N,M-N} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} \\ I_{M-N,M-N} \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} \mathbf{0} & I_{M-N,M-N} \end{bmatrix}}_{\Phi^*}, \tag{15}$$

and

$$A = \Phi^* F_M = \begin{bmatrix} \mathbf{0} & I_{M-N,M-N} \end{bmatrix} F_M. \tag{16}$$

Therefore, for each J with $|J| = M - N$, $[A]_{\bullet,J}$ is a Vandermonde matrix pre-multiplied by an invertible diagonal matrix, and is full rank. Thus by Theorem 1, the harmonic frame is robust to $M - N$ erasures.

Since $|[F_M]_{i,j}| = 1/\sqrt{M}$ for all $i, j = 1, 2, \cdots, M$, we can easily check that $[A^*A]_{i,i} = \frac{M-N}{M}$. Hence the harmonic frame is a ENPTF by Theorem 2.

References

[1] V.K. Goyal, J. Kovačević, and J.A. Kelner. Quantized Frame Expansions with Erasures. *Applied and Computational Harmonic Analysis*, 10(3):203–233, 2001.

[2] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2002.

[3] P.G. Casazza and J. Kovačević. Equal-Norm Tight Frames with Erasures. *Advances in Computational Mathematics*, 18(2):387–430, 2003.