MATH 247 — FALL 2000 — TEST 3 SOLUTIONS

NAME:

Total: 100 points. Do 4 out of 5 questions. You **MUST** do #5. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

1 (25 points)**.** Deduce all solutions x of:

$$\begin{aligned} & x \equiv 1 \mod 3 \\ \text{and} \quad & x \equiv 2 \mod 4. \end{aligned}$$

Guessing is not acceptable: you should use the method of the Chinese Remainder Theorem, or some similar technique.

Solution 1:

Since $x \equiv 1 \mod 3$, we have $x = 1 + 3k$ for some $k \in \mathbb{Z}$. But $x \equiv 2 \mod 4$ and so $1 + 3k \equiv 2 \mod 4$, or $3k \equiv 1 \mod 4$. Multiplying by 3 gives $9k \equiv 3 \mod 4$, which reduces to $k \equiv 3 \mod 4$ (since $9k \equiv k \mod 4$). That is, $k = 3 + 4\ell$ for some $\ell \in \mathbb{Z}$.

Thus $x = 1 + 3k = 10 + 12k\ell$. So x equals 10 plus a multiple of 12. The set of all solutions is $\{x = 10 + 12m : m \in \mathbb{Z}\}$.

Solution 2:

Following the proof of the Chinese Remainder Theorem, we set

$$\begin{aligned} N &= 3 \cdot 4 = 12, \\ N_1 &= 4, \\ N_2 &= 3. \end{aligned}$$

We want to find solutions of

$$\begin{aligned} N_1 y_1 &\equiv 1 \mod 3, \qquad \text{that is,} \quad 4y_1 \equiv 1 \mod 3, \\ N_2 y_2 &\equiv 1 \mod 4, \qquad \text{that is,} \quad 3y_2 \equiv 1 \mod 4. \end{aligned}$$

We see that $y_1 = 1$ and $y_2 = 3$ will do. So then one solution is

$$x = a_1 N_1 y_1 + a_2 N_2 y_2 = 1 \cdot 4 \cdot 1 + 2 \cdot 3 \cdot 3 = 22.$$

The Chinese Remainder Theorem tells us the solutions are hence the numbers of the form $x = 22 + 12m$ for $m \in \mathbb{Z}$ (using here that $N = 12$).

2 (25 points). [Do not simplify your answers below, or evaluate any binomial coefficients.] There are two boxes, each containing a huge number of colored balls:

in Box 1, 70% of the balls are orange and 30% are blue;

in Box 2, 30% of the balls are orange and 70% are blue.

(a) You reach into Box 1 and randomly take out 12 balls. Explain why the probability of getting 8 orange balls and 4 blue balls is $p_1 = \binom{12}{8}(0.7)^8(0.3)^4$.

Solution: Each possible ordering of choosing 8 orange balls out of 12 will occur with probability $(0.7)^8(0.3)^4$, because on each choice, you get an orange ball with probability 0.7 and a blue ball with probability 0.3. Further, there are $\binom{12}{8}$ different ways to choose the 8 orange balls out of 12.

(b) You reach into Box 2 and randomly take out 12 balls. Explain why the probability of getting 8 orange balls and 4 blue balls is $p_2 = \binom{12}{8}(0.3)^8(0.7)^4$.

Solution: Each possible ordering of choosing 8 orange balls out of 12 will occur with probability $(0.3)^8(0.7)^4$, because on each choice, you get an orange ball with probability 0.3 and a blue ball with probability 0.7. Further, there are $\binom{12}{8}$ different ways to choose the 8 orange balls out of 12.

(c) Finally, you reach into a box (not knowing which box it is) and randomly take out 12 balls. You get 8 orange balls and 4 blue balls. Find the probability that you reached into Box 1.

Solution:

The conditional probability of having reached into Box 1 given that you got 8 orange balls and 4 blue balls is:

$$P(\text{you reached into Box 1}|\text{you got 8 orange balls and 4 blue balls})$$

$$=\frac{P(\text{you reached into Box 1 and you got 8 orange balls and 4 blue balls})}{P(\text{you got 8 orange balls and 4 blue balls})}$$

$$=\frac{\frac{1}{2}p_1}{\frac{1}{2}p_1+\frac{1}{2}p_2} \qquad \text{since you have probability } \frac{1}{2} \text{ of reaching into each of Box 1 and Box 2}$$

$$=\frac{\frac{1}{2}\binom{12}{8}(0.7)^8(0.3)^4}{\frac{1}{2}\binom{12}{8}(0.7)^8(0.3)^4+\frac{1}{2}\binom{12}{8}(0.3)^8(0.7)^4} = \frac{(0.7)^4}{(0.7)^4+(0.3)^4} \approx 96.7\%.$$

3 (25 points)**.** Fix $n, k \in \mathbb{N}$. Suppose that n pairs of socks are put into the laundry, with each sock having one mate. The laundry machine randomly eats socks; a random set of k socks returns. Determine the expected number of complete pairs of returned socks.
Hints:
1. Here S is the set of all k-element subsets of the total set of $2n$ socks. It is assumed that each outcome is equally likely.
2. For each $i = 1, \ldots, n$, let X_i be a random variable on S that equals 1 if both socks in the i-th pair are returned, and 0 otherwise. The question is asking you to find $E(X_1 + \; + X_n)$.
3. Use the linearity of expectation.
4. Show $E(X_i) = P(X_i = 1)$ the probability that the i-th pair is returned.
5. Evaluate $P(X_i = 1)$.

Solution:

Let S be the possibility space, consisting of k-sock subsets of the total $2n$ socks. This space has $\binom{2n}{k}$ elements, each occuring with probability $1/\binom{2n}{k}$.

Let X be the number of pairs returned. We wish to calculate $E(X)$.

Number the pairs 1 through n, and let X_i be the random variable on S which is 1 if the i-th pair is returned, and 0 otherwise. Then the number of pairs returned is

$$X = X_1 + \cdots + X_n,$$

and so

$$E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$$

by the linearity of expectation.

The number of k-sock sets that contain the i-th pair is

$$\binom{2n-2}{k-2},$$

because after choosing the 2 socks from the i-th pair, we still have to choose another $k-2$ socks from the remaining $2n-2$ socks. Thus the probability of getting the i-th pair returned is

$$\frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}.$$

Now, the expected value of X_i is

$$\begin{aligned} E(X_i) &= 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) \\ &= P(X_i = 1) \\ &= \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}, \end{aligned}$$

and so the expected value of X is

$$E(X) = n \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}.$$

4 (25 points)**.**

(a) Prove: "Suppose $a_n \to L$ and $b_n \to M$, and that $a_n \leq b_n$ for all n. Then $L \leq M$."

(b) Disprove, by finding a counterexample: "Suppose $a_n \to L$ and $b_n \to M$, and that $a_n < b_n$ for all n. Then $L < M$."

Solution:

(a) Let $c_n = a_n - b_n$, so that $c_n \leq 0$ for all n. Then $\lim c_n \leq 0$, by Lemma 13.17. But $\lim c_n = \lim a_n - \lim b_n = L - M$, so $L - M \leq 0$. That is, $L \leq M$.

Or: Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ for all $n \geq N$. Then

$$\begin{aligned} L - \varepsilon &< a_n \\ &\leq b_n < M + \varepsilon \qquad \text{for all } n \geq N. \end{aligned}$$

This implies $L - \varepsilon < M + \varepsilon$, so that $L - M < 2\varepsilon$. This holds for all $\varepsilon > 0$, and so necessarily $L - M \leq 0$. That is, $L \leq M$.

Or: [Contradiction] Suppose instead that $L > M$. Let $\varepsilon = \frac{L-M}{2}$, so that $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ for all $n \geq N$. Then for all $n \geq N$,

$$\begin{aligned} a_n &> L - \varepsilon \\ &= \frac{L+M}{2} \\ &= M + \varepsilon \\ &> b_n. \end{aligned}$$

But this contradicts the assumption that $a_n \leq b_n$ for all n. Hence actually $L \leq M$.

(b) Let $a_n = \frac{1}{n}$ and $b_n = \frac{2}{n}$. Then $a_n < b_n$ for all n but $L = M$ since: $L = \lim a_n = 0$ and $M = \lim b_n = 0$.

5 (25 points). Prove the following statement (which is part of the **Monotone Convergence Theorem**):

"Every bounded nondecreasing sequence converges to its supremum."

Solution:

Let $\langle a \rangle$ be a bounded nondecreasing sequence. Since $\langle a \rangle$ is bounded, it has an upper bound. Hence it has a supremum, by Completeness of $\mathbb{R}$. Write $\alpha = \sup\langle a \rangle$ for this supremum.

We wish to show $a_n \to \alpha$. Let $\varepsilon > 0$. Notice $\alpha - \varepsilon$ is smaller than the least upper bound α, and so it is not an upper bound for the sequence. Thus there exists some term of the sequence that is bigger than $\alpha - \varepsilon$, say $a_N > \alpha - \varepsilon$.

Since the sequence $\langle a \rangle$ is nondecreasing, we know $a_n \geq a_N$ for all $n \geq N$. Hence $a_n > \alpha - \varepsilon$ for all $n \geq N$. But also $a_n \leq \alpha$ for all n, because α is the supremum. Hence

$$-\varepsilon < a_n - \alpha \leq 0 \qquad \text{for all } n \geq N.$$

This implies $|a_n - \alpha| < \varepsilon$ for all $n \geq N$, and so $a_n \to \alpha$ as desired.