MATH 247 — FALL 2000 — TEST 3

NAME:

Total: 100 points. Do 4 out of 5 questions. You **MUST** do #5. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

1 (25 points)**.** Deduce all solutions x of:

$$\begin{aligned} x &\equiv 1 \mod 3 \\ \text{and} \quad x &\equiv 2 \mod 4. \end{aligned}$$

Guessing is not acceptable: you should use the method of the Chinese Remainder Theorem, or some similar technique.

2 (25 points). [Do not simplify your answers below, or evaluate any binomial coefficients.] There are two boxes, each containing a huge number of colored balls:

in Box 1, 70% of the balls are orange and 30% are blue;

in Box 2, 30% of the balls are orange and 70% are blue.

(a) You reach into Box 1 and randomly take out 12 balls. Explain why the probability of getting 8 orange balls and 4 blue balls is: $p_1 = \binom{12}{8}(0.7)^8(0.3)^4$.

(b) You reach into Box 2 and randomly take out 12 balls. Explain why the probability of getting 8 orange balls and 4 blue balls is: $p_2 = \binom{12}{8}(0.3)^8(0.7)^4$.

(c) Finally, you reach into a box (not knowing which box it is) and randomly take out 12 balls. You get 8 orange balls and 4 blue balls. Find the probability that you reached into Box 1.

3 (25 points)**.** Fix $n, k \in \mathbb{N}$. Suppose that n pairs of socks are put into the laundry, with each sock having one mate. The laundry machine randomly eats socks; a random set of k socks returns. Determine the expected number of complete pairs of returned socks.

Hints:

1. Here S is the set of all k-element subsets of the total set of $2n$ socks. It is assumed that each outcome is equally likely.
2. For each $i = 1, \ldots, n$, let X_i be a random variable on S that equals 1 if both socks in the i-th pair are returned, and 0 otherwise. The question is asking you to find $E(X_1 + \; + X_n)$.
3. Use the linearity of expectation.
4. Show $E(X_i) = P(X_i = 1)$ the probability that the i-th pair is returned.
5. Evaluate $P(X_i = 1)$.

4 (25 points)**.**

(a) Prove: "Suppose $a_n \to L$ and $b_n \to M$, and that $a_n \leq b_n$ for all n. Then $L \leq M$."

(b) Disprove, by finding a counterexample: "Suppose $a_n \to L$ and $b_n \to M$, and that $a_n < b_n$ for all n. Then $L < M$."

5 (25 points)**.** Prove the following statement (which is part of the **Monotone Convergence Theorem**):

"Every bounded nondecreasing sequence converges to its supremum."