MATH 247 — FALL 2000 — TEST 2 SOLUTIONS

NAME:

Total: 100 points. Do 5 out of 6 questions. You **MUST** do #6. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

1 (20 points)**.**
(a) [15 points] Prove without using calculus that the following function is a bijection:

$$f : (0,1) \to (0,\infty)$$
$$f(x) = \frac{1}{x} - 1$$

(b) [5 points] Using part (a), construct a bijection $g : (-1,1) \to \mathbb{R}$ (so that the interval $(-1,1)$ has the same cardinality as $\mathbb{R}$).

Solution:

(a) We have to show that for each $y \in (0,\infty)$ there exists exactly one $x \in (0,1)$ with $f(x) = y$. So let $y \in (0,\infty)$. Then

$$y = f(x) = \frac{1}{x} - 1$$

is equivalent to

$$y + 1 = \frac{1}{x}, \qquad \text{or} \qquad x = \frac{1}{y+1}.$$

[Since $y > 0$ we know $y + 1 > 1$, and so we can certainly divide by $y + 1$.] Obviously $x > 0$, and also $x < 1$ since $y + 1 > 1$. Hence $x \in (0,1)$. Since this is the unique $x \in (0,1)$ with $f(x) = y$, we are done.

(b)
Define

$$g(x) = \begin{cases} f(x) = \frac{1}{x} - 1 & x \in (0,1) \\ -f(-x) = \frac{1}{x} + 1 & x \in (-1,0) \\ 0 & x = 0. \end{cases}$$

Sketch the graph of this function!

By part (a), g is a bijection from $(0,1)$ to $(0,\infty)$, and by symmetry g is a bijection from $(-1,0)$ to $(-\infty,0)$. Since also $g(0) = 0$, g is a bijection from $(-1,1)$ to $(-\infty,\infty)$ as desired.

2 (20 points)**.** Suppose that $S = \{a_1, a_2, a_3, \ldots\}$ is a countable set. Explain why each infinite subset of S is also countable.

Solution:

Let $T \subseteq S$ be an infinite subset of S. List the elements of T in increasing order as

$$T = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$$

with $n_1 < n_2 < n_3 < \cdots$. The fact that T can be written as an infinite list shows T is countable.

[For example, to write the subset $T = \{a_2, a_4, a_6, \ldots\}$ in the above way, we would have $n_1 = 2, n_2 = 4, n_3 = 6, \ldots$.]

In words, just imagine listing the elements of $S = \{a_1, a_2, a_3, \ldots\}$, except only write down the elements of S that belong to the subset T.

3 (20 points)**.** Let $[n]$ be the set $\{1, \ldots, n\}$. Show that

$$\sum_{A \subseteq [n]} \sum_{B \subseteq [n]} |A \cap B| = n(2^{n-1})^2.$$

Hint: count in two ways the set of triples (x, A, B), where $A, B \subseteq [n]$ and $x \in A \cap B$.

Solution:

For a particular pair of subsets $A, B \subseteq [n]$, the number of triples of the form (x, A, B) with $x \in A \cap B$ is just the number of elements in $A \cap B$, that is $|A \cap B|$. It follows that the number of triples (x, A, B) as A and B range over all subsets of $[n]$ is given by the left side of the above formula.

On the other hand, we can construct all such triples by first choosing the element x, and then constructing sets $A, B \subset [n]$ which contain x. There are n choices for the element x, then 2^{n-1} choices for the subset A (a choice of "in or out" for the remaining $n - 1$ elements of $[n]$ that might belong to A, along with x), then similarly 2^{n-1} choices for the subset B. This gives

$$n(2^{n-1})^2 = n4^{n-1}$$

for the number of triples, which is the right side of the formula.

4 (20 points)**.** Consider a standard 52-card deck.
(a) [4 points] How many different 5-card hands are there? [You need not evaluate the binomial coefficient in your answer.]
(b) [12 points] How many 5-card hands have at least three cards with the same rank?
(c) [4 points] What is the *probability* that a random 5-card hand has at least three cards with the same rank?

Solution:

(a)

The number of ways of choosing 5 cards from 52 is $\binom{52}{5}$.

(b)

This is

$$\text{(nr of hands with 3 cards of same rank)} + \text{(nr of hands with 4 cards of same rank)} =$$
$$\text{(nr of choices of rank)(nr of choices of 3 cards of that rank)(nr of choices for remaining 2 cards)}+$$
$$\text{(nr of choices of rank)(nr of choices of 4 cards of that rank)(nr of choices for remaining card)} =$$
$$\binom{13}{1}\binom{4}{3}\binom{48}{2} + \binom{13}{1}\binom{4}{4}\binom{48}{1}.$$

(c)

The probability is

$$\frac{\text{answer to (b)}}{\text{answer to (a)}} = \frac{\binom{13}{1}\binom{4}{3}\binom{48}{2} + \binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

5 (20 points)**.** Let $k \in \mathbb{N}$. Prove that if $\sqrt[3]{k}$ is a rational number then $k = m^3$ for some $m \in \mathbb{N}$.

Hint: use the prime factorizations of one or more numbers.

Solution:

Suppose $\sqrt[3]{k}$ is a rational number, and write it as $\sqrt[3]{k} = \frac{a}{b}$ in lowest terms (meaning a and b have no common factor bigger than 1). We can assume a and b are both positive (since the cube root of a positive number is positive).

Then $kb^3 = a^3$. We show $b = 1$. For suppose instead that $b > 1$. Let p be a prime factor of b. Then $p|kb^3$ and so $p|a^3$. Hence $p|a$, by Proposition 6.7. So a and b have a common factor bigger than 1, namely p. Contradiction! Hence $b = 1$, and so $\sqrt[3]{k} = a \in \mathbb{N}$ as desired.

Note: several other proofs are possible.

6 (20 points)**.** Let $n \in \mathbb{N}$. Remember we call integers x and y **congruent modulo** n if $x - y$ is divisible by n, and we write $x \equiv y \mod n$.
(a) [10 points] Prove that congruence modulo n is an equivalence relation on $\mathbb{Z}$.
(b) [10 points] Prove using modular arithmetic that $n^5 + 4n$ is a multiple of 5.

Solution:

(a)
We must show that for integers a, b, c, congruence modulo n is

(1) Reflexive: $a \equiv a$
(2) Symmetric: if $a \equiv b$ then $b \equiv a$
(3) Transitive: if $a \equiv b$ and $b \equiv c$ then $a \equiv c$

Reflexive: $a - a = 0$ is divisible by n.
Symmetric: If $a - b = nk$ then $b - a = n(-k)$.
Transitive: If $a - b = nk$ and $b - c = n\ell$ then $a - c = (a - b) + (b - c) = n(k + \ell)$.

(b) What must be shown is that $n^5 + 4n \equiv 0 \mod 5$. We only need to check the numbers $n = 0, 1, 2, 3, 4$, since we are working modulo 5.

$n = 0$: $n^5 + 4n \equiv 0 \mod 5$.
$n = 1$: $n^5 + 4n \equiv 5 \equiv 0 \mod 5$.
$n = 2$: $n^5 + 4n \equiv 40 \equiv 0 \mod 5$.
$n = 3$: $n^5 + 4n \equiv 3 \cdot 9 \cdot 9 + 4 \cdot 3 \equiv 3 \cdot 4 \cdot 4 + 2 \equiv 50 \equiv 0 \mod 5$.
$n = 4$: $n^5 + 4n \equiv 4 \cdot 16 \cdot 16 + 4 \cdot 4 \equiv 4 \cdot 1 \cdot 1 + 1 \equiv 5 \equiv 0 \mod 5$.

Alternative solution: Since 5 is prime, $n^5 \equiv n \mod 5$ by Fermat's Little Theorem and so

$$n^5 + 4n \equiv n + 4n \equiv 5n \equiv 0 \mod 5.$$