**MATH 247 — FALL 2000 — TEST 2**

NAME:

Total: 100 points. Do 5 out of 6 questions. You **MUST** do #6. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

**1** (20 points)**.**
(a) [15 points] Prove without using calculus that the following function is a bijection:

$$f : (0, 1) \to (0, \infty)$$
$$f(x) = \frac{1}{x} - 1$$

(b) [5 points] Using part (a), construct a bijection $g : (-1, 1) \to \mathbb{R}$ (so that the interval $(-1, 1)$ has the same cardinality as $\mathbb{R}$).

**2** (20 points)**.** Suppose that $S = \{a_1, a_2, a_3, \ldots\}$ is a countable set. Explain why each infinite subset of $S$ is also countable.

**3** (20 points)**.** Let $[n]$ be the set $\{1, \ldots, n\}$. Show that

$$\sum_{A \subseteq [n]} \sum_{B \subseteq [n]} |A \cap B| = n(2^{n-1})^2.$$

Hint: count in two ways the set of triples $(x, A, B)$, where $A, B \subseteq [n]$ and $x \in A \cap B$.

**4** (20 points)**.** Consider a standard 52-card deck.
(a) [4 points] How many different 5-card hands are there? [You need not evaluate the binomial coefficient in your answer.]
(b) [12 points] How many 5-card hands have at least three cards with the same rank?
(c) [4 points] What is the *probability* that a random 5-card hand has at least three cards with the same rank?

**5** (20 points)**.** Let $k \in \mathbb{N}$. Prove that if $\sqrt[3]{k}$ is a rational number then $k = m^3$ for some $m \in \mathbb{N}$.

Hint: use the prime factorizations of one or more numbers.

**6** (20 points)**.** Let $n \in \mathbb{N}$. Remember we call integers $x$ and $y$ **congruent modulo** $n$ if $x - y$ is divisible by $n$, and we write $x \equiv y \mod n$.
(a) [10 points] Prove that congruence modulo $n$ is an equivalence relation on $\mathbb{Z}$.
(b) [10 points] Prove using modular arithmetic that $n^5 + 4n$ is a multiple of 5.