Solutions

MATH 247 — FALL 2000 — TEST 1

NAME:

Total: 100 points. Do 5 out of 6 questions. You **MUST** do #6. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

1 (20 points)**.** Let $S = \{x \in \mathbb{R} : x^2 > 2x + 8\}$ and $T = \{x \in \mathbb{R} : x > 4\}$. Are the following statements true or false?

(a) $T \subseteq S$

(b) $S \subseteq T$

Solution: The key to the problem is to factor the quadratic:

$$x^2 - 2x - 8 = (x - 4)(x + 2).$$

Using this idea, we see that

$$\begin{aligned} S &= \{x \in \mathbb{R} : x^2 > 2x + 8\} \\ &= \{x \in \mathbb{R} : x^2 - 2x - 8 > 0\} \\ &= \{x \in \mathbb{R} : (x - 4)(x + 2) > 0\} \\ &= \{x \in \mathbb{R} : x > 4 \text{ or } x < -2\}. \end{aligned}$$

So $T \subseteq S$ is true, because every number $x > 4$ does belong to S.

But $S \subseteq T$ is false, since the numbers $x < -2$ belong to S but not to T. [For example, $-3 \in S$ because $(-3)^2 > 2(-3) + 8$. But $-3 \notin T$.]

2 (20 points)**.** Without using words of negation (such as "no", "not",...), write the negations of the following statements.

a) For all real numbers A there is an $x < A$ such that $f(x) > B$.

b) There exists $c \in \mathbb{R}$ such that for all real numbers $x, y \geq c$, if $x > y$ then $f(x) > f(y)$.

Solution:

a) In symbols, part a) says

$$(\forall A \in \mathbb{R})(\exists x < A) f(x) > B.$$

To negate this, we change $\forall$ to $\exists$, change $\exists$ to $\forall$, and negate the statement at the end, getting:

$$(\exists A \in \mathbb{R})(\forall x < A) \neg (f(x) > B).$$

But $\neg(f(x) > B)$ means $f(x) \leq B$, and so the desired negation is

$$(\exists A \in \mathbb{R})(\forall x < A) f(x) \leq B.$$

In words, one could write:

For some real number A, for all $x < A$ we have $f(x) \leq B$.

b) In symbols, part b) says

$$(\exists c \in \mathbb{R})(\forall x, y \geq c)(x > y \Longrightarrow f(x) > f(y)).$$

To negate this, we change $\forall$ to $\exists$, change $\exists$ to $\forall$, and negate the statement at the end, getting:

$$(\forall c \in \mathbb{R})(\exists x, y \geq c) \neg (x > y \Longrightarrow f(x) > f(y)).$$

But if it is *not* true that $(x > y \Longrightarrow f(x) > f(y))$, then we must have $x > y$ and $f(x) \leq f(y)$. Hence the desired negation is

$$(\forall c \in \mathbb{R})(\exists x, y \geq c)(x > y \text{ and } f(x) \leq f(y)).$$

In words, one could write:

For each $c \in \mathbb{R}$ there exists $x, y \geq c$ such that $x > y$ and $f(x) \leq f(y)$.

3 (20 points)**.** Let

$$f(x) = \frac{x^2 - 1}{x^2 + 4}, \qquad x \in \mathbb{R}.$$

Show that the image of f is $[-\frac{1}{4}, 1)$.

Solution: Write $f(\mathbb{R})$ for the image of f.

i) $f(\mathbb{R}) \subseteq [-\frac{1}{4}, 1)$.

Proof: For every $x \in \mathbb{R}$, we have

$$f(x) = \frac{x^2 - 1}{x^2 + 4} < 1 \qquad \text{because } x^2 - 1 < x^2 + 4$$

(this last inequality simplifies to the true inequality $-1 < 4$). And also

$$f(x) = \frac{x^2 - 1}{x^2 + 4} \geq -\frac{1}{4} \qquad \text{because } x^2 - 1 \geq -\frac{1}{4}(x^2 + 4)$$

(this last inequality simplifies to the true inequality $\frac{5}{4}x^2 \geq 0$).

This proves that $-\frac{1}{4} \leq f(x) < 1$ for all x, and so $f(\mathbb{R}) \subseteq [-\frac{1}{4}, 1)$.

ii) $f(\mathbb{R}) \supseteq [-\frac{1}{4}, 1)$.

Proof: Let $y \in [-\frac{1}{4}, 1)$. We want to show there exists an $x \in \mathbb{R}$ with $f(x) = y$, because then we will know $y \in f(\mathbb{R})$. In fact we can find two such x-values:

$$\begin{aligned} f(x) = y \quad &\iff \quad \frac{x^2 - 1}{x^2 + 4} = y \\ &\iff \quad x^2 - 1 = y(x^2 + 4) \\ &\iff \quad (1 - y)x^2 = 1 + 4y \\ &\iff \quad x^2 = \frac{1 + 4y}{1 - y} \\ &\iff \quad x = \pm\sqrt{\frac{1 + 4y}{1 - y}}. \end{aligned}$$

It is OK to take the square root here because the number inside is nonnegative: $1 + 4y \geq 0$ because $y \geq -\frac{1}{4}$, and $1 - y > 0$ because $y < 1$.

This completes the proof that $f(\mathbb{R}) = [-\frac{1}{4}, 1)$.

4 (20 points)**.** Let $n \geq 3$. Prove by induction that every set of n elements has $\frac{1}{2}n(n-1)$ subsets of size 2.

[For example, the set $A = \{x_1, x_2, x_3\}$ has the following subsets of size two: $\{x_1, x_2\}, \{x_1, x_3\}$ and $\{x_2, x_3\}$. Here $n = 3$, and notice $\frac{1}{2}n(n-1) = 3$, which correctly gives the number of subsets of size two. This proves your induction basis.]

Solution:

Write $P(n)$ for the statement that "every set of n elements has $\mathbf{\frac{1}{2}n(n-1)}$ subsets of size 2".

(a) [Basis step] Let $n = 3$. Then $P(3)$ is true, as shown in the statement of the problem.

(b) [Induction step] Assume $P(n)$ is true for $n = k$, so that every set of k elements has $\frac{1}{2}k(k-1)$ subsets of size 2. Let A be a set of $k+1$ elements. To complete the induction step, we need to show that A has

$$\frac{1}{2}(k+1)k$$

subsets of size 2, since that is the statement $P(n)$ with $n = k+1$.

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$ for our set of $k+1$ elements. Obviously there are $\mathbf{k}$ subsets of size 2 that contain x_{k+1}, namely the subsets $\{x_1, x_{k+1}\}, \{x_2, x_{k+1}\}, \ldots, \{x_k, x_{k+1}\}$. And the subsets of size 2 that do *not* contain x_{k+1} are precisely the subsets of $\{x_1, x_2, \ldots, x_k\}$ of size 2. There are $\mathbf{\frac{1}{2}k(k-1)}$ such subsets by the induction hypothesis. Adding up, we find the number of subsets of A of size 2 is:

$$\frac{1}{2}k(k-1) + k = \frac{1}{2}k(k-1+2) = \frac{1}{2}(k+1)k,$$

exactly as we needed to show.

5 (20 points)**.** For $n \geq 2$, find and prove a formula for $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)$.

Solution: First we investigate numerically:

$n = 2 \implies \prod_{i=2}^{n}(1-\frac{1}{i^2}) = (1-\frac{1}{2^2}) = \frac{3}{4}$

$n = 3 \implies \prod_{i=2}^{n}(1-\frac{1}{i^2}) = (1-\frac{1}{2^2})(1-\frac{1}{3^2}) = \frac{2}{3}$

$n = 4 \implies \prod_{i=2}^{n}(1-\frac{1}{i^2}) = (1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2}) = \frac{5}{8}$

$n = 5 \implies \prod_{i=2}^{n}(1-\frac{1}{i^2}) = (1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2}) = \frac{3}{5}$

So the results for $n = 2, 3, 4, 5$ are $\frac{3}{4}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}$. There is no really obvious pattern here. But wait! $\frac{2}{3} = \frac{4}{6}$ and $\frac{3}{5} = \frac{6}{10}$. So the list can be rewritten as $\frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \frac{6}{10}$. Now we see that the n^{th} term matches up with $\frac{n+1}{2n}$, at least for the cases $n = 2, 3, 4, 5$.

Now we prove our conjecture that

$$\prod_{i=2}^{n}(1-\frac{1}{i^2}) = \frac{n+1}{2n} \qquad \text{for all } n \geq 2.$$

Proof.

(a) [Basis step] The first case to consider here is $n = 2$ (notice we don't consider $n = 1$ in this problem). We already did the basis step above: when $n = 2$ we have $\prod_{i=2}^{n}(1-\frac{1}{i^2}) = (1-\frac{1}{2^2}) = \frac{3}{4} = \frac{n+1}{2n}$.

(b) [Induction step] Let $k \geq 2$. Assume the formula holds for $n = k$, so that

$$\prod_{i=2}^{k}(1-\frac{1}{i^2}) = \frac{k+1}{2k}.$$

Then

$$\begin{aligned}
\prod_{i=2}^{k+1}(1-\frac{1}{i^2}) &= \left[1-\frac{1}{(k+1)^2}\right]\prod_{i=2}^{k}(1-\frac{1}{i^2}) \quad \text{by splitting of the last term of the product} \\
&= \left[1-\frac{1}{(k+1)^2}\right]\frac{k+1}{2k} \qquad \text{by the induction hypothesis} \\
&= \frac{[(k+1)^2-1]}{(k+1)^2}\frac{k+1}{2k} \\
&= \frac{k^2+2k}{2k(k+1)} \\
&= \frac{k+2}{2(k+1)},
\end{aligned}$$

which is the desired formula with $n = k + 1$. This proves the induction step. □

6 (20 points). [**You MUST do this problem.**] Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

(i) Prove that $f(0) = 0$.
(ii) Let $s \in \mathbb{R}$. Prove by induction that $f(ns) = nf(s)$ for all $n \in \mathbb{N}$.
(iii) Let $t \in \mathbb{R}$. Deduce using part (ii) that $f(\frac{m}{n}t) = \frac{m}{n}f(t)$ for all $m, n \in \mathbb{N}$.

Solution:

(i) Choosing $x = 0, y = 0$, the given formula becomes

$$f(0+0) = f(0) + f(0).$$

That is, $f(0) = f(0)+f(0)$. Subtracting $f(0)$ from both sides of this equation gives $0 = f(0)$, as needed.

(ii) Write $P(n)$ for the statement that $f(ns) = nf(s)$.
[Basis step] With $n = 1$ the statement $P(1)$ says $f(1s) = 1f(s)$, or $f(s) = f(s)$, which is true.
[Induction step] Assume $P(n)$ is true for $n = k$, so that $f(ks) = kf(s)$. Then

$$\begin{aligned} f((k+1)s) &= f(ks+s) \\ &= f(ks) + f(s) && \text{by the given equation applied with } x = ks, y = s \\ &= kf(s) + f(s) && \text{by our induction hypothesis} \\ &= (k+1)f(s). \end{aligned}$$

That is, we have proved $P(n)$ is true with $n = k+1$, completing the induction proof.

(iii) We have

$$\begin{aligned} nf(\frac{m}{n}t) &= f(n\frac{m}{n}t) && \text{by applying part (ii) with } s = \frac{m}{n}t \\ &= f(mt) \\ &= mf(t) && \text{by applying part (ii) with } s = t. \end{aligned}$$

Dividing through by n gives $f(\frac{m}{n}t) = \frac{m}{n}f(t)$, as we wanted.