MATH 247 — FALL 2000 — TEST 1

NAME:

Total: 100 points. Do 5 out of 6 questions. You **MUST** do #6. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

1 (20 points)**.** Let $S = \{x \in \mathbb{R} : x^2 > 2x + 8\}$ and $T = \{x \in \mathbb{R} : x > 4\}$. Are the following statements true or false?
(a) $T \subseteq S$
(b) $S \subseteq T$

2 (20 points)**.** Without using words of negation (such as "no", "not",...), write the negations of the following statements.

a) For all real numbers A there is an $x < A$ such that $f(x) > B$.

b) There exists $c \in \mathbb{R}$ such that for all real numbers $x, y \geq c$, if $x > y$ then $f(x) > f(y)$.

3 (20 points)**.** Let

$$f(x) = \frac{x^2 - 1}{x^2 + 4}, \qquad x \in \mathbb{R}.$$

Show that the image of f is $[-\frac{1}{4}, 1)$.

4 (20 points)**.** Let $n \geq 3$. Pove by induction that every set of n elements has $\frac{1}{2}n(n-1)$ subsets of size 2.

[For example, the set $A = \{x_1, x_2, x_3\}$ has the following subsets of size two: $\{x_1, x_2\}, \{x_1, x_3\}$ and $\{x_2, x_3\}$. Here $n = 3$, and notice $\frac{1}{2}n(n-1) = 3$, which correctly gives the number of subsets of size two. This proves your induction basis.]

5 (20 points)**.** For $n \geq 2$, find and prove a formula for $\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right)$.

6 (20 points). [**You MUST do this problem.**] Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

(i) Prove that $f(0) = 0$.

(ii) Let $s \in \mathbb{R}$. Prove by induction that $f(ns) = nf(s)$ for all $n \in \mathbb{N}$.

(iii) Let $t \in \mathbb{R}$. Deduce using part (ii) that $f(\frac{m}{n}t) = \frac{m}{n}f(t)$ for all $m, n \in \mathbb{N}$.