MATH 247 — FALL 2000 — FINAL EXAM — BRIEF SOLUTIONS

NAME:

Total: 200 points. Do 8 out of 12 questions. **You MUST indicate which 8 questions are to be graded; otherwise, just the first 8 problems will be graded.** EXPLAIN every answer. No books, notes, calculators or computers allowed on this exam.

1 (25 points)**.**

(a) [8 points] A function $f(x)$ on $[a, b]$ is called *bounded* if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Negate this, so obtaining the definition of an *unbounded* function.

Solution. For all $M \in \mathbb{R}$ there exists $x \in [a, b]$ such that $|f(x)| > M$.

(b) [8 points] Define what it means to say that "a_n converges to L".

Solution. For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - L| < \varepsilon$.

(c) [9 points] Negate your answer in part (b), thus obtaining a definition of "it is false that a_n converges to L".

Solution. There exists $\varepsilon > 0$ such that for each $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - L| \geq \varepsilon$.

2 (25 points). Consider a function $f : \mathbb{Z} \to \mathbb{R}$ such that $f(1) = 2$, $f(m) > 0$ for all $m \in \mathbb{Z}$, and

$$f(j-k) = \frac{f(j)}{f(k)} \qquad \text{for all } j, k \in \mathbb{Z}.$$

Using these properties, find a formula for $f(m), m \in \mathbb{Z}$. (Hint: play around to guess a formula, and then use induction ideas to give a proper proof.)

Solution. The formula $f(j-k) = f(j)/f(k)$ looks like the law of exponents, and so we guess $f(m) = a^m$ for some a. Then since $f(1) = 2$ we guess $a = 2$.

Our guess is $f(m) = 2^m$ for $m \in \mathbb{Z}$. Prove this result by induction in two directions:

[Basis] $f(1) = 2 = 2^1$ and so the guess is correct for $m = 1$.

[Induction upwards] If $f(m) = 2^m$ then $f(m+1) = f(m)f(1)$ by the given formula with $j = m+1, k = 1$, and hence $f(m+1) = 2^m 2 = 2^{m+1}$.

We have proved the guess for all $m \geq 1$, by Induction. Now argue similarly for $m \leq 1$, by Induction downwards (do it!).

3 (25 points).

(a) [15 points] Prove that if $n \in \mathbb{N}$ and $q \geq 2$ then $n < q^n$.

Solution. Proposition 3.16.

(b) [10 points] Prove $\mathbb{N} \times \mathbb{N}$ is countable.

Solution. Theorem 4.44 (and also draw a suitable diagram showing the lattice points and the diagonals through them, to make the proof more understandable).

4 (25 points)**.** Prove that with repetition allowed, there are $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$ ways to select n objects from k types.

Solution. Theorem 5.23.

5 (25 points)**.** Let p be a prime number.

(a) [12 points] Prove that p divides $\left(\begin{array}{c} p \\ k \end{array}\right)$ if $1 \leq k \leq p-1$. (Hint: count the number of ways to choose a k-person subcommittee with a chair, from a p-person committee.)

Solution. Exercise 6.37a. (Homework 7 #7, in Spring 2009.)

(b) [13 points] Prove that $n^p - n$ is divisible by p, for every $n \in \mathbb{N}$. (Hint: try induction, making use of the Binomial Theorem and also part (a) of this question.)

Solution. This is Exercise 6.37b. Just follow the hint: by the Binomial Expansion,

$$\begin{aligned}(n+1)^p - (n+1) &= \sum_{k=0}^{p} \binom{p}{k} n^k 1^{p-k} - n - 1 \\ &= \sum_{k=1}^{p-1} \binom{p}{k} n^k 1^{p-k} + (n^p - n).\end{aligned}$$

Now use part (a) and induction on n.

A quicker solution is to use the Corollary 7.39 that follows from Fermat's Little Theorem.

6 (25 points).

(a) [8 points] Prove that if $a \equiv r \mod n$ and $b \equiv s \mod n$, then $a + b \equiv r + s \mod n$ and $a \cdot b \equiv r \cdot s \mod n$.

Solution. Lemma 7.19.

(b) [17 points] [Chinese Remainder Theorem] Prove that if $\{n_i\}$ is a set of r natural numbers that are pairwise relatively prime, and $\{a_i\}$ are any r integers, then the system of congruences $x \equiv a_i \mod n_i$ has a unique solution modulo $N = \prod_i n_i$.

Solution. Theorem 7.30.

7 (25 points).

(a) [10 points] Show directly that $(5, 12, 13)$ is a Pythagorean triple, and then show that it has one of the forms

$$(2rs, r^2 - s^2, r^2 + s^2), \quad (r^2 - s^2, 2rs, r^2 + s^2),$$

or possibly a multiple of one of these forms.

Solution. type (ii): $r = 3, s = 2$

(b) [15 points] Fix $c \in \mathbb{Z}$, and define $f(x) = x^6 + cx^5 + 1$. Show that if $c \neq \pm 2$ then f has no rational zeros. Does f have rational zeros when $c = \pm 2$?

Solution. Here $c_0 = 1$ and $c_6 = 1$. By the Rational Zeros Theorem, if $x = p/q$ is a rational zero in lowest terms then $p | c_0$ and $q | c_6$, and so $p = \pm 1, q = 1$. Thus the only possible rational zeros are $x = p/q = \pm 1$. We have

$$f(\pm 1) = 1 \pm c + 1 = 2 \pm c.$$

So if $c \neq \pm 2$ then $f(\pm 1) \neq 0$ and so f has no rational zeros.

If $c = +2$ then $f(-1) = 0$, so $x = -1$ is a rational zero.

If $c = -2$ then $f(+1) = 0$, so $x = +1$ is a rational zero.

8 (25 points).

(a) [8 points] Prove that if $|b_n - L| \leq a_n$ for all n and $a_n \to 0$, then $b_n \to L$.

Solution. Proposition 13.12.

(b) [17 points] Prove that every convergent sequence is a Cauchy sequence.

Solution. Proposition 14.13.

9 (25 points)**.** Use the definition of limit to prove that $\lim[(a_n - 4)^{-1}] = \frac{1}{3}$, if $\lim a_n = 7$.

Solution. Very similar to Exercise 13.26 (on Homework 10 in Spring 2009).

10 (25 points). NOT COVERED in Spring 2009.

(a) [8 points] Consider f defined on a deleted neighborhood of a. State the definition of what

$$\lim_{x \to a} f(x) = L$$

means.

Solution. See Definition 15.4.

(b) [8 points] Define what it means for a function f to be continuous at a. Define what it means for f to be continuous on the interval (c, d).

Solution. See Definition 15.10.

(c) [9 points] Give an example of a function f that is defined at every point $x \in [0, 2]$ but that is *not* continuous at $x = 1$; *prove* that it is not continuous at $x = 1$.

Solution. For example, let

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 3 & \text{for } 1 \leq x \leq 2. \end{cases}$$

Here $\lim_{x \to 1} f(x)$ does not exist (because the limit from the left equals 0, while the limit from the right equals 3; supply a rigorous proof!). Hence in particular, $\lim_{x \to 1} f(x) \neq f(1)$, so that f is not continuous at $x = 1$.

11 (25 points)**.** [Intermediate Value Theorem] NOT COVERED in Spring 2009.

Prove that if f is continuous on $[a, b]$, and $f(a) < y < f(b)$, then there exists an $x \in (a, b)$ such that $f(x) = y$.

Solution. Theorem 15.19.

12 (25 points)**.** NOT COVERED in Spring 2009. Prove that if f is continuous on $[a, b]$, then f is bounded.

Solution. Theorem 15.24.