MATH 247 — FALL 2000 — FINAL EXAM

NAME:

Total: 200 points. Do 8 out of 12 questions. **You MUST indicate which 8 questions are to be graded; otherwise, just the first 8 problems will be graded.**
EXPLAIN every answer. No books, notes, calculators or computers allowed on this exam.

1 (25 points)**.**
(a) [8 points] A function $f(x)$ on $[a, b]$ is called *bounded* if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Negate this, so obtaining the definition of an *unbounded* function.
(b) [8 points] Define what it means to say that "a_n converges to L".
(c) [9 points] Negate your answer in part (b), thus obtaining a definition of "it is false that a_n converges to L".

2 (25 points)**.** Consider a function $f : \mathbb{Z} \to \mathbb{R}$ such that $f(1) = 2$, $f(m) > 0$ for all $m \in \mathbb{Z}$, and

$$f(j-k) = \frac{f(j)}{f(k)} \qquad \text{for all } j, k \in \mathbb{Z}.$$

Using these properties, find a formula for $f(m), m \in \mathbb{Z}$. (Hint: play around to guess a formula, and then use induction ideas to give a proper proof.)

3 (25 points).

(a) [15 points] Prove that if $n \in \mathbb{N}$ and $q \geq 2$ then $n < q^n$.

(b) [10 points] Prove $\mathbb{N} \times \mathbb{N}$ is countable.

4 (25 points)**.** Prove that with repetition allowed, there are $\left(\begin{array}{c} n+k-1 \\ k-1 \end{array} \right)$ ways to select n objects from k types.

5 (25 points)**.** Let p be a prime number.

(a) [12 points] Prove that p divides $\binom{p}{k}$ if $1 \leq k \leq p-1$. (Hint: count the number of ways to choose a k-person subcommittee with a chair, from a p-person committee.)

(b) [13 points] Prove that $n^p - n$ is divisible by p, for every $n \in \mathbb{N}$. (Hint: try induction, making use of the Binomial Theorem and also part (a) of this question.)

6 (25 points)**.**

(a) [8 points] Prove that if $a \equiv r \mod n$ and $b \equiv s \mod n$, then $a + b \equiv r + s \mod n$ and $a \cdot b \equiv r \cdot s \mod n$.

(b) [17 points] [Chinese Remainder Theorem] Prove that if $\{n_i\}$ is a set of r natural numbers that are pairwise relatively prime, and $\{a_i\}$ are any r integers, then the system of congruences $x \equiv a_i \mod n_i$ has a unique solution modulo $N = \prod_i n_i$.

7 (25 points).

(a) [10 points] Show directly that $(5, 12, 13)$ is a Pythagorean triple, and then show that it has one of the forms

$$(2rs, r^2 - s^2, r^2 + s^2), \quad (r^2 - s^2, 2rs, r^2 + s^2),$$

or possibly a multiple of one of these forms.

(b) [15 points] Fix $c \in \mathbb{Z}$, and define $f(x) = x^6 + cx^5 + 1$. Show that if $c \neq \pm 2$ then f has no rational zeros. Does f have rational zeros when $c = \pm 2$?

8 (25 points).

(b) [8 points] Prove that if $|b_n - L| \leq a_n$ for all n and $a_n \to 0$, then $b_n \to L$.

(b) [17 points] Prove that every convergent sequence is a Cauchy sequence.

9 (25 points)**.** Use the definition of limit to prove that $\lim[(a_n - 4)^{-1}] = \frac{1}{3}$, if $\lim a_n = 7$.

10 (25 points)**.**

(a) [8 points] Consider f defined on a deleted neighborhood of a. State the definition of what

$$\lim_{x \to a} f(x) = L$$

means.

(b) [8 points] Define what it means for a function f to be continuous at a. Define what it means for f to be continuous on the interval (c, d).

(c) [9 points] Give an example of a function f that is defined at every point $x \in [0, 2]$ but that is *not* continuous at $x = 1$; *prove* that it is not continuous at $x = 1$.

11 (25 points). [Intermediate Value Theorem]

Prove that if f is continuous on $[a, b]$, and $f(a) < y < f(b)$, then there exists an $x \in (a, b)$ such that $f(x) = y$.

12 (25 points). Prove that if f is continuous on $[a, b]$, then f is bounded.