Four Color Fest
Activity Book

Written and illustrated by Melinda Lanius
Department of Mathematics
University of Illinois at Urbana-Champaign
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To Matt Ando, Michelle Delcourt, and Philipp Hieronymi,
for joining me in my love of sharing research-level mathematics with children
and diligently finding the funds to make it happen.

Foreword: Dear parents, teachers, and activity facilitators, before beginning each activity with your child or students, please carefully read the directions for each activity. While each section is designed to be reminiscent of a typical activity book, almost always I’ve altered the standard rules to illustrate a particular mathematical concept. If you are curious about the abstract mathematics underpinning these activities, more information can be found in an accompanying set of lesson plans on my webpage: www.math.uiuc.edu/~lanius2/outreach.html If you have any comments, questions, or concerns, please feel free to write to me at lanius2@illinois.edu

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Directions.
As you visit stations and play games, collect stamps on your treasure map.

At the end of your visit, redeem your map for a prize at Treasure Trove!
Our story begins over two hundred years ago in the city of Königsberg, Prussia. The city straddles the Pregel river and includes two large islands. The islands are connected to the banks of the river by seven bridges.

**Riddle:** Can you walk through the city of Königsberg and cross each of these seven bridges once and only once?

The citizens of Königsberg wondered whether they could leave home, cross each bridge exactly once, and return home. Unable to answer their question, they posed their riddle to mathematician Leonhard Euler.

Euler solved the riddle: it is impossible to leave home, cross each bridge exactly once, and return home. In arguing his solution, he created an area of mathematics called graph theory. Mathematicians still study and use graph theory to this day.

The year 2017 marks the 40 year anniversary of the Four Color theorem, a result in graph theory that was proven at the University of Illinois.
In 1852, a mathematics student Francis Guthrie was trying to color a map of English counties. He noticed that he only needed 4 different colors so that touching counties were not the same color. He wondered, 

*can any map be colored only using 4 colors?*

Francis did not know, so he asked his brother Frederick. But Frederick did not know, so he asked his teacher, mathematician Augustus De Morgan. Augustus also did not know! So he wrote to his friend, mathematician Sir William Rowan Hamilton:

*A student of mine asked me to day to give him a reason for a fact which I did not know was a fact-and do not yet. He says that if a figure can be any how divided and the compartments differently coloured so that the figures with any common boundary line are differently coloured-four colours may be wanted, but not more...*

In the end, Sir William also did not know if four colors can color a map. It took over 100 years for someone to prove that Francis’ statement was true! On June 21, 1976 University of Illinois professors Kenneth Appel and Wolfgang Haken announced that they had a proof. They had reduced proving the theorem to checking 1,936 maps one-by-one. This seemed an impossible task to do themselves, so they taught a computer how to do the checking. Even then, it took the computer over a thousand hours! This is the first major theorem that has not been proven without the help of a computer.

The games in this book will show us how Appel and Haken’s proof used graph theory and computers to prove the four color theorem.
A graph is a collection of dots, called vertices, that are connected by not-necessarily-straight lines, called edges.

Example: Consider the following graph. The vertices are the dots labeled with numbers 1, 2, 3, and 4.

There is an edge from 1 to 4, an edge from 2 to 3, from 2 to 4, and from 3 to 4. Note that the place where two edges cross is not a vertex.

Directions.

Let’s finish drawing the graph. This graph has vertices 1, 2, and 3.

The edge connecting 1 to 2 and the edge connecting 1 to 4 are already drawn. Can you finish drawing the graph by connecting 1 to 3?

Edges:
- 1 to 2
- 1 to 3
- 1 to 4

Graph:

Directions.

Draw the graph:
There are vertices 1, 2, 3, 4, 5, and 6.
The edges are listed.

Edges:
- 1 to 2
- 1 to 3
- 2 to 3
- 3 to 4
- 3 to 6
- 4 to 5
- 5 to 6

Graph:
Directions. Draw the graph.

Edges:

- □ 1 to 3
- □ 1 to 2
- □ 2 to 13
- □ 3 to 6
- □ 4 to 5
- □ 4 to 9
- □ 5 to 10
- □ 6 to 7
- □ 6 to 11
- □ 7 to 8
- □ 8 to 12
- □ 8 to 14
- □ 9 to 20
- □ 10 to 12
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- □ 80 to 82
- □ 81 to 83
- □ 82 to 83

Graph:
GRAPH ISOMORPHISM

Two graphs that contain the same number of vertices connected in the same way are the same graph. We call them isomorphic graphs.

Example: The three graphs are the same. To see that, we check that the graphs have the same vertices. They all have 1, 2, 3, 4, and 5.

We then check that they have the same edges.

Edges:
- □ 1 to 2,
- □ 2 to 5,
- □ 5 to 4,
- □ 4 to 3,
- □ 3 to 1.

Graphs:

Directions. Circle the two graphs that are the same. Note that the numbering of the vertices does not change.

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A graph is *planar* if it can be drawn without crossing edges.

**Example:** These graphs are the same. We obtain the graph on the right from the left by moving vertex 1. Since the one on the right is drawn without crossing edges, the graph is planar.

**Directions.** Untangle the graph to show that it is planar. You can rearrange the vertices as well as draw the edges with not-necessarily-straight lines.
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A torus is a hollow donut. There are some graphs that cannot be drawn on a sheet of paper (i.e. plane) without crossing edges, but they can be drawn on a donut (i.e. torus) without crossing edges.

**Example:** This graph cannot be drawn without crossing edges on the plane. We’ve shown how it can be drawn on a torus without crossing edges.

**Directions.** Decorate the donuts! If you could decorate a donut, what would it look like?
MAP COLORING

A map is a separation of a plane into regions. We want to find a way to color the regions of any map so that no two adjacent regions have the same color. Two regions are called adjacent if they share a common boundary that is not a corner.

Example: Valid colorings of a map of the Four Corners region of the United States.

Non-example: An invalid coloring of a map of Four Corners.

Directions. Using only 4 colors, color each square so that if two shapes share an edge, then they are different colors.
Directions. Using only 4 colors, color the map of the state Telangana in India. Remember, if two regions share a border, then they should be different colors.

Telangana, India

Are 4 colors always necessary to color a map? Can you color the shield below using only 3 colors? Why or why not?
How many colors are necessary to color the flower? Color the flower using this many colors.
For each map, we can build a planar graph. Each region is given a vertex. If two regions share a border, then we put an edge between their vertices:

Directions. For the map given, draw a graph model.
Map: Champaign County, Illinois 1881
Directions. For the map given, draw a graph model.

Map: Counties of Massachusetts

Mathematical Modeling. Now to study maps, we can just study planar graphs. Every map gives us a planar graph.
We want our rules for coloring maps to translate to coloring the vertices of the graph. If regions shared a border in a map, we said they cannot be colored the same. For the graph, this means if two vertices share an edge, then they must be different colors.

**Example:** A valid coloring of a map corresponds to a valid coloring of the associated graph.

**Directions.** Color each graph with the fewest number of colors so that no vertices of the same color are connected by an edge.
Directions. Color the graph using at most 4 colors so that no vertices of the same color are connected by an edge.
The 4 color theorem tells us that every planar map can be colored with four colors. We have seen some maps that need 4. How can we tell when 4 are needed, or if three would suffice? This is a very hard question to answer, what is called a NP-complete problem.

One way we can determine if a graph needs 4 colors is to look for a particular subgraph, or configuration hidden inside the graph, that we know needs 4 colors. For instance, the graph on the left below requires 4 colors since every vertex shares an edge with every other vertex:

It can be very hard to find copies of this subgraph because they may not be drawn in the same way. For instance the two graphs above are the same graph, drawn differently.

*Directions.* See how many copies of the subgraph you can find in the graph.

**WILL THREE SUFFICE?**
The graph needs 4 colors because every vertex is connected to every other vertex by an edge. However, there are some graphs that require 4 colors even though this graph is not a subgraph. For instance, the following graph requires 4 colors so that no vertices of the same color share an edge:

Directions. See how many copies of the subgraph with 6 vertices you can find in the graph.
Directions. To solve a sudoku puzzle, place a number into each box so that each row across, each column down, and each small 9-box square within the larger diagram (there are 9 of these) will contain every number from 1 through 9. In other words, no number will appear more than once in any row, column, or smaller 9-box square. Working with the numbers already given as a guide, complete each diagram with the missing numbers that will lead to the correct solution.

SOLVING SUDOKU VIA GRAPH COLORING
The graph used to find a solution to a 9 by 9 Sudoku grid has 81 vertices and is too large to draw easily, so we will work through the case of a 4 by 4 grid. The Sudoku rules are analogous to the 9 by 9 case: You place a number into each box so that each row across, each column down, and each small 4-box square within the larger diagram (there are 4 of these) will contain every number from 1 through 4.

We associate a graph to the Sudoku board by putting a vertex in each cell (so there are 16 vertices), together with edges as follows: two vertices are connected by an edge if the cells that they correspond to are in the same column, row, or small 4-box square. We have thus represented the Sudoku grid as a graph.

This graph is a tool we can use to solve a Sudoku puzzle. We use the given numbers to partially color the associated graph. We assign each of the numbers 1 through 4 a color. If the number \(i\) appears in a cell, we color the corresponding vertex with \(i\)'s color. If a cell is empty, we don’t color the corresponding vertex. There is exactly one way to complete the coloring of the graph so that no vertices sharing an edge have the same color. This coloring gives us the solution to the puzzle.
Game theory.
In a game, perfect play is the behavior or strategy of a player that leads to the best possible outcome for that player regardless of the response by the opponent. A solved game is a game whose outcome can be correctly predicted from any position, assuming that both players play perfectly.

Directions. Play a few games of tic-tac-toe with a friend. What are some good strategies for winning? Is tic-tac-toe a solved game?

Classical Tic-Tac-Toe:
Two players, X and O, take turns marking one of the spaces in a three by three grid. The first player to place three of their marks in a horizontal, vertical, or diagonal row wins.

TIC-TAC-TOE ON GRAPHS
Building a graph from a board.

We can associate a graph to a tic-tac-toe board.

Each space becomes a vertex. Two vertices are connected by an edge if they are next to one another in the board.

Graph Tic-Tac-Toe Directions:

Two players, X and O, take turns marking the vertices of a graph. The first player to place three of their marks on vertices $x$, $y$, and $z$ where there is an edge from $x$ to $y$ and from $y$ to $z$ wins.

Example: We can see how the moves on the classical board correspond to moves on the associated graph.
Directions. Play a few games of graph tic-tac-toe with a friend. With perfect play, does the first player always win?

Directions. Try tic-tac-toe a more complicated graph. With perfect play, is either player guaranteed to win?
Directions. Sometimes adding one more vertex to a graph can make a huge difference in perfect play. Play tic-tac-toe on both graph 1 and 2. Compare the strategies between these two graphs.

Graph 1.

Graph 2.

Directions. In the boxes below, draw your own graph tic-tac-toe boards and challenge a friend to play!
An *algorithm* is a list of steps to follow to solve a problem. We write algorithms so that computers can help us do our jobs. Our steps must be very clear because the computer will only do what we tell it.

**Example.** I have a mixed pile of apples and oranges.

I want the robot Gere to sort the fruit into two piles: apples together and oranges together. An algorithm is a list of steps that Gere can follow to sort the fruit.

**Directions.** Below there are three lists of steps for Gere to follow. Only one of them is an algorithm that Gere can follow to successfully sort the fruit. Decide which one works and explain why the other two do not work.

**List 1**
Step 1: Pick up a fruit from the pile.
Step 2: Label it as an apple or an orange.

**List 2**
Step 1: Pick up a fruit from the pile.
Step 2: If an apple, set to the side. If an orange, return to the pile.
Step 3: Return to step 1.

**List 3**
Step 1: Pick up a fruit from the pile.
Step 2: If an apple, set to the left. If an orange, set to the right.
Step 3: If there is still fruit in the pile, return to step 1. Otherwise, stop.

**Hint:** By following the steps in a list, does Gere know what to do with the fruit once he has identified it as an apple or an orange? Does Gere know when he has finished sorting the fruit and can stop?
A maze is a collection of paths with an entrance and an exit. Some of the paths result in dead-ends! Your job is to find your way from the entrance to the exit. We can also use algorithms to solve mazes!

**The wall follower rule.** Keep one hand in contact with one wall of the maze. You will not get lost and will reach the other exit.

**Dead-end filling.** First, find all of the dead-ends in the maze, and then “fill in” the path from each dead-end until the first junction is met. Repeat this process until no dead-ends remain. Note that some passages won’t become parts of a dead-end until other dead-ends are filled in.

**Directions.** Use the wall follower rule to solve the maze.

**Directions.** Use dead-end filling to solve the maze.
Directions. Use an algorithm to solve the mazes.

Maze Credits: mazegenerator.net
WORD SEARCH ALGORITHMS

A word search is a word game that consists of the letters of words placed in a grid. The objective is to find and mark all the words hidden inside the box. The words may be placed horizontally, vertically, or diagonally.

Directions. Write an algorithm that the computer Talus can follow to complete a word search. Use your algorithm to complete the word search.

Hint: Where should Talus start scanning? Which direction should Talus then look: down, across, or diagonally? What do you want Talus to do once they've found a word? If you don't give them a new step, they'll keep scanning for the word they've already found!

Algorithm: ____________________________________________________________

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Directions. Use your algorithm to complete the word searches.
Can you change your algorithm so you can complete the puzzles faster?

Word bank: red yellow blue
green orange violet indigo

Word bank: 000101 101101 011111 111100
010011 011110 110001 100101 011101
**Knight’s Tour**

**Super Bonus Challenge Problem.** A *knight’s tour* is a sequence of moves of a knight on a chessboard where the knight visits every square only once.

Can you find a knight’s tour on the standard 8 by 8 chessboard?

There are quite a number of ways to find a knight’s tour on a given board with a computer. Some of these methods are algorithms while others are heuristics.

Can you think of an algorithm to solve the riddle? A brute-force search (checking all possible sequences of moves) for a knight’s tour is impractical because there are

\[ 4 \times 10^{51} \]

possible move sequences! This is well beyond the capacity of modern computers. With some human inspiration, computers can in fact find a knight’s tour.

Can you think of a way to turn this problem into a problem about graphs? What graph could model the chess board and the knight’s possible moves? A knight’s tour would correspond to what in the graph?

The knight has eight possible moves from its current square.
Select Solutions

Connect the dots:

Sudoku:

<table>
<thead>
<tr>
<th>Easy</th>
<th>Medium</th>
<th>Hard</th>
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</thead>
<tbody>
<tr>
<td><img src="easy_sudoku_solution.png" alt="Easy Sudoku Solution" /></td>
<td><img src="medium_sudoku_solution.png" alt="Medium Sudoku Solution" /></td>
<td><img src="hard_sudoku_solution.png" alt="Hard Sudoku Solution" /></td>
</tr>
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</table>

Graph tic-tac-toe:

For [ ], the first player will always win with perfect play.

For [ ], the game will always be a draw with perfect play.

For [ ], the first player will always win with perfect play.

For [ ], the game will always be a draw with perfect play.

Algorithms: In List 1, Gere doesn’t know what to do with the fruit once he has identified it, so this is not an algorithm for sorting fruit into 2 piles. In List 2, Gere does not know when he has successfully sorted the fruit. After the apples have all been removed from the pile, he will pick up oranges and place them back for forever. List 3 is an algorithm for sorting the fruit into two piles.