1. Introduction

In recent years, the Ricci flow has proved itself –by means of G. Perelman’s implementation of R. Hamilton’s program– a phenomenal tool in studying the geometry/topology of three-manifolds. In higher dimensions one of the main stumbling blocks—which still eludes many an approach– is the understanding of the formation of finite time singularities. The standard approach to try and analyze such singularities is through blow-up analysis, but this seems to yet be unapproachable in dimension higher than three, due to the lack of various pinching phenomena (such as Hamilton-Ivey’s in dimension three).

Nonetheless, for compact Kähler manifolds, since the Ricci flow equation reduces to scalar parabolic equation, much has been said about the behavior of the solutions. To fix the notation, let us write the Ricci flow equation on a compact Kähler manifold $(X, J, \omega_0)$ in the form:

\[
\begin{aligned}
\frac{\partial \tilde{\omega}}{\partial t} &= - \text{Ric}(\tilde{\omega}) \\
\tilde{\omega}(0) &= \omega_0
\end{aligned}
\] (1)
One of the first most striking results in this direction has been a theorem due to Tian and Zhang (cf. [Tian-Zhang] and [Tian08]) which asserts that on a compact Kähler manifold, the Ricci flow will exists on an interval \([0, T_{\text{max}}]\) where:

\[ T_{\text{max}} := \sup \{ t : \text{there is a representative } \alpha \in [\omega_0] - t \cdot c_1(X), \text{ satisfying } \alpha > 0 \} \]

that is to say, it will exist as long as the obvious topological restrictions allow it, in that there are holomorphic curves whose area with respect to \(\tilde{\omega}(t)\) goes to zero as \(t \to T_{\text{max}}\), so that one leaves the Kähler cone passed \(T_{\text{max}}\) (this theorem was also proved by [Cascini-La Nave] in the projective case).

In fact, the second named author (cf. [Tian08]) showed that the solution \(\tilde{\omega}_t\) to the Kähler-Ricci flow (eq. (1))—at finite times (i.e., \(T_{\text{max}} < +\infty\))—converges as \(t \to T := T_{\text{max}}\) to a \((1,1)\)-current \(\tilde{\omega}_T\). Conjecturally such a current is smooth away from some analytic subvariety (this was proved in [Tian-Zhang] in case \(([\omega_0] - c_1(X))^k > 0\)).

For finite time singularities, the authors (cf. [La Nave-Tian09]) have initiated a program that allows a geometric picture of the needed surgeries via variations of Kähler-Ricci flow:

\[
\begin{aligned}
\frac{\partial \tilde{\omega}}{\partial t} &= -\text{Ric}(\tilde{\omega}) + \lambda \tilde{\omega} \\
\tilde{\omega}(0) &= \omega_0
\end{aligned}
\]

and the solutions of a soliton-type equation, named V-soliton equation, (up to suitably re-normalizing the moment map coordinate \(\tau\)):

\[
\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} \log |V|_\varphi^2 = \lambda \omega
\]

does exist with initial value \(\omega_0\) is a family of Kähler manifolds \((X_i, \tilde{\omega}(t))\) with \(\dim(X_i) \geq 0\) together with a sequence \(T_0 = 0 < T_1 < T_2 < \cdots < T_k < \cdots T'\), called surgery times, such that:

1. For any time \(t \in [T_i, T_{i+1})\), the complex structure \(J_t\) is constant on \(X_t\); for \(t = 0\), \(X_0\) coincides with \(X\).
2. For each \(i\), \(\dim X_i \geq \dim X_{i+1}\) and there is a Kähler manifold \(Y_i\) and holomorphic maps \(p_i : X_i \to Y_i\) and \(p_{i+1} : X_{i+1} \to Y_i\). In the projective case \(X_{i+1}\) is the flip.
3. For \(t \in [T_i, T_{i+1})\) the Kähler form \(\omega(t)\) is a solution of (1) (possibly with singularities), either with initial Kähler metric \(\omega_0\) or \(\tilde{\omega}(t) = \tilde{\omega}_0 + \sqrt{-1} \partial \bar{\partial} u(t)\) and \(p_{i+1}^* p_i^* \omega_{T_i}\) has potential \(\tilde{u}_{T_i}\) and:

\[
\lim_{t \to T_{i}^+} \|u(t) - u_{T_i}\|_{L^\infty} = 0
\]
Technically speaking, this definition should just be considered provisional, as one should need to request that $X_i$ and $X_{i+1}$ be isomorphic away from tubular neighborhoods of analytic subsets $\Sigma_i \subset X_i$ and $\Sigma_{i+1} \subset X_{i+1}$ and furthermore that the singular set $S_i$ of the flow $\omega(t)$ with $t \in [T_i, T_{i+1}]$ be contained in $\Sigma_t$.

In our context, these "surgeries" correspond to when the moment map parameter $\tau$ goes through a critical value of the moment map–via the correspondence $\tau(t) = Ct$ (here $C > 0$ is an explicit constant)– where, if $\tau_i := \tau(T_i)$, we simply take $X_i := \mu^{-1}(\tau_i - \epsilon)/S_1$ and $X_{i+1} := \mu^{-1}(\tau_i + \epsilon)/S_1$.

In [La Nave-Tian09] the authors showed the equivalence, in the compact case, of eq. (3) and a scalar equation –named scalar $V$-soliton equation– which has the form:

$$\frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} u)^n}{\omega_0^n} = |V|^2 e^{F-\lambda u}$$

where $|V|^2 e^{F-\lambda u} = |V|^2_0 + \sqrt{-1} \partial \bar{\partial} u(V, JV)$ and $|V|^2_0 = \omega_0(JV, V)$ or, alternatively, $|V|^2_0 = |V|^2_{g_0}$ where $g_0(x, y) = \omega_0(Jx, y)$ is the Riemannian metric associated to $\omega_0$.

In this note, we address the regularity of equation (4) on a compact manifold. Such an equation is a degenerate elliptic equation of Monge-Ampere type. It differs considerably from Calabi/Yau’s Monge-Ampere equation because of the term $|V|^2_u$ on the right side, which depends on the second derivatives of the unknown.

In what follows we will assume the existence of a holomorphic Hamiltonian action of $S^1$ on $(M, \omega)$ and we we denote by $Z$ the holomorphic vector field associated the the action Our main result is (cf. Theorem 6.3, Lemma 6.1 and Theorem 95):

**Main Theorem.** Let $(M, g_0)$ be a compact Kähler manifold of complex dimension $n$ endowed with a a holomorphic Hamiltonian action of $S^1$ on $(M, \omega)$ generated by a holomorphic vector field $Z$. Then, there exists a constant $C$, depending only on $(M, g_0)$, $V$ and $\|F\|$, such that, for any $x \in M$:

$$0 < \langle n - 1 + \Delta^H u \rangle(x) < C.$$  \hfill (5)

In particular,

$$0 < \sup_M |\nabla^H \nabla^H u| < C.$$

Furthermore, for any compact set $K \subset M \setminus \{x \in M : V(x) = 0\}$, there exists a constant $C = C(K, \kappa, ||F||_{C^{2,\alpha}})$ depending only on $||F||_{C^{2,\alpha}}$, $\kappa := \max_M ||\nabla^H \nabla^H u||$ and $v := \frac{1}{||V||_{C^{2,\alpha}(K)}}$ such that:

$$\sup_K ||\nabla u|| < C.$$
We now outline an argument that shows that in the projective case, when the Ricci flow develops finite time singularities and it is non-collapsing, (and if \( \omega_0 \) is a rational class) the metric completion of \((X \setminus S, \omega(T))\) is actually \((X_0, \omega(T))\), assuming a result analogous to the Main Theorem, but for manifolds with boundary. This result was stated with a sketch of a proof in [La Nave-Tian09] and recently proved in [Song-Weinkove].

Suppose one does get to a singular time \( T := T_i \), and consider \( p^- := p_i : X^- := X_i \to Y = Y_i \) and \( p^+ := p_{i+1} : X^+ := X_{i+1} \to Y := Y_i \) as above. We assume that there is a Kähler manifold (possibly with boundary) with a Hamiltonian \( S^1 \) action and moment map \( \mu : M \to \mathbb{R} \), such that (at least in a neighborhood of the singular set \( S \)) \( X^- = \mu^{-1}(-\epsilon)/S^1 \) and \( X^+ = \mu^{-1}(\epsilon)/S^1 \) (this assumption is satisfied for all toric birational transformations, which include blow-ups). Since we are in the projective case, the results of [Song-Tian] apply and we may find a (generalized) solution \( u \) to the scalar equation associate to eq. (1) on \((T - \delta, T) \times X^- \cup (T, T + \delta) \times X^+ \) satisfying:

\[
\lim_{t \to T^-} ||u(t) - p^- u_T||_{L^\infty} = 0 \quad \text{and} \quad \lim_{t \to T^+} ||u(t) - p^+ u_T||_{L^\infty} = 0.
\]

with \( u_T \in L^\infty(Y) \). By assumption there is a neighborhood of the singular set \( S \) in \((T - \delta, T) \times X^- \cup X^+ \cup (T, T + \delta) \times X^+ \) which can be described as a \( M/S^1 \). Let \( \pi : M \to M/S^1 \) be the natural projection. Now the solution \( u \) can be lifted to a bounded solution of \( \phi \) of the V-soliton equation on \( M \setminus \pi^{-1}(S) \). Now find a solution of the V-soliton equation \( \varphi \) (with boundary if \( \partial M \neq \emptyset \), in which case \( \phi \big|_{\partial M} \in C^\infty(\partial M) \)) on \( M \). By the maximum principle this must coincide with \( \phi \) and therefore the Kähler-Ricci flow must have the regularity dictated by our Main Theorem. Now it is easy to see that the Gromov-Hausdorff limit of \((X, \tilde{\omega}(t))\) as \( t \to T \) (assuming this is a finite time collapsing singularity) must be \((Y, \omega_T)\).

Alternatively, in the previous discussion one could avoid using the results of [Song-Tian], which rely on deep results of algebraic geometry, by solving the free boundary problem of the V-soliton equation, where one assigns the boundary value just at \( \mu^{-1}(\tau(0)) \).

2. Scalar V-soliton equation

2.1. Preliminaries. Let \((M, g_0)\) be a compact Kähler manifold with a Hamiltonian \( S^1 \)-action by isometries. Let \( V \) be its associated Killing field. Then \( V \) is the imaginary part of a holomorphic vector field \( Z = i^{1/2}(JV + \sqrt{-1}TV) \) on \( M \), where \( J \) denotes the complex structure on \( M \).

In this section, we will study the solvability of the following complex Monge-Ampère equation:

\[
(\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n = \left( |V|^2_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u(V, JV) \right) e^F \omega_{g_0}^n,
\]

where \( F \) is a given function satisfying

\[
\int_M (|V|^2_{g_0} e^F - 1) \omega_{g_0}^n = 0.
\]
Lemma 2.1. There is a uniform constant $C = C(g_0)$ such that for any smooth $S^1$-invariant function $u$ such that $\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u \geq 0$, we have

$$|J V(u)| \leq C.$$ 

We will use the perturbation method to solve (6) for a $S^1$-invariant $u$. Consider

$$(\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n = \left( \epsilon + |V|^2_{g_0} \right) e^{F_\epsilon} \omega^n_{g_0},$$

where $\epsilon > 0$, $\omega_\epsilon = \omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u$ and $F_\epsilon = F + c_\epsilon$ for a constant $c_\epsilon$ satisfying:

$$\int_M \left( (\epsilon + |V|^2_{g_0}) e^{F_\epsilon} - 1 \right) \omega^n_{g_0} = 0.$$

Now let us introduce some notations. Set

$$C^{k,\alpha}(M, V) := \left\{ u \in C^{k,\alpha}(M) \mid V(u) = 0 \right\},$$

where $C^{k,\alpha}(M)$ is the Hölder space of $C^k$-smooth functions such that

$$||u||_{C^{k,\alpha}} := \sum_{i=1}^k \sup_{x \in M} |\nabla^i u| + \sup_{x,y \in M, x \neq y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{d(x, y)^\alpha} < +\infty,$$

where $d(\cdot, \cdot)$ denotes the distance function of any fixed metric $g$. Clearly, this coincides with the space $C^{k,\alpha}(M)_{S^1}$ which consists of $S^1$-invariant functions in $C^{k,\alpha}(M)$. We further set

$$C^{k,\alpha}(M; V)_g := \left\{ v \in C^{k,\alpha}(M; V) \mid \int_M \left( (\epsilon + |V|^2_{g_0}) e^v - 1 \right) \omega^n_{g_0} = 0 \right\}$$

and

$$C^{k,\alpha}_g(M; V) := \left\{ u \in C^{k,\alpha}(M; V) \mid \int_M u \omega^n_{g_0} = 0 \right\}.$$ 

For $k \geq 2$, we also denote by $P^{k,\alpha}(M, V)$ the set of all $u \in C^{k,\alpha}(M, V)$ such that $\omega_u := \omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u > 0$. Define a differential operator from $P^{k,\alpha}(M, V)$:

$$\Phi_\epsilon(u) := \log \left( \frac{(\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n}{\omega^n_{g_0}} \right) - \log(\epsilon + |V|^2_u),$$

where $|V|^2_u = \omega_u(V, J V)$ is the square norm of $V$ with respect to the metric given by $\omega_u$.

Clearly, for $k \geq 2$, $\Phi_\epsilon$ maps into $C^{k-2,\alpha}_{g_0}(M; V)$. To solve (7), we only need to show that $\Phi_\epsilon$ is surjective. We will prove that for $k$ sufficiently large,$^2$

$$\Phi_\epsilon(P^{k,\alpha}(M, V) \cap C^{k,\alpha}_{g_0}(M; V)) = C^{k-2,\alpha}_{g_0}(M; V).$$

$^2k \geq 4$ should be sufficient.
The tangent space to $C^{k-2,\alpha}(M;V)_{g_0}$ at $\Phi_\varepsilon(u)$ is the space: $C^{k-2,\alpha}_g(M;V)$. Hence, the differential $D\Phi_\varepsilon|u$ of $\Phi_\varepsilon$ at $u$ is a linear map from $C^{k,\alpha}(M;V)$ into $C^{k-2,\alpha}_g(M;V)$. Furthermore, we recall from [La Nave-Tian09] Lemma 2.2. For any $\varepsilon > 0$, $\Phi_\varepsilon$ is an elliptic operator. Moreover, for any $u \in P^{k,\alpha}(M,V)$, the differential $D\Phi_\varepsilon|u$ is surjective with only constant functions in its kernel.

As it was mentioned earlier, we will use the continuity method to solve (7). Fix a large $k > 0$. Choose any path $F_{\varepsilon,s}$ in $C^{k-2,\alpha}(M;V)_{g_0}$ ($s \in [0,1]$) with $F_{\varepsilon,0} = -\log(\varepsilon + |V|^2_{g_0})$ and $F_{\varepsilon,1}$ coincides with $F_\varepsilon$ in (7). Consider a family of complex Monge-Ampère equations:

$$(\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n = (\varepsilon + |V|^2_{g_0}) e^{F_{\varepsilon,s}} \omega_{g_0}^n, \quad (8)$$

where $\omega_g = \omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u$. Define

$I = \{ s \in [0,1] \mid (8) \text{ has a solution for any } s' \in [0,s] \}$. Clearly, $0 \in I$ since $u = 0$ is a solution. It follows from the above lemma and the Inverse Function Theorem

**Corollary 2.3.** The set $I$ defined above is open.

Hence, to establish the existence of a solution for (7) is equivalent to proving that $I$ is closed. For this purpose, we need a prior estimates for solutions of (8). The $C^0$-estimate has been established in [La Nave-Tian09] and is stated as follows:

**Proposition 2.4.** There is a uniform constant $C$ which depends only on $(M, g_0)$, $||F||_{C^1(M)}$, $\sup_M |V|_{g_0}$ and $\sup_M |\text{div}(JV)|$ such that for any solution $u$ of (8) with $\int_M u \omega_{g_0}^n = 0$, we have

$$\sup_M |u| \leq C.$$ 

**2.2. The symplectic picture.** Here we recall a few facts from [La Nave-Tian09] Let $\mathbb{C}^*$ act holomorphically and Hamiltonially on a Kähler manifold $M$ of dimension $n$. Let $m := n - 1$. Let $z_1, \cdots, z_m$ be holomorphic coordinates on the quotient manifold $X_\alpha$, and let $\tau$ be “moment map coordinates”, i.e., $\mu = \tau$—where $\mu : M \to \text{Lie}(S^1)^* = \mathbb{R}$ is the moment map of the action—sometime, we simply identify $\mu$ with its value $\tau$. Clearly, we have $d\tau = i_V \omega$, where $\{V\}$ is a vector field generating the Hamiltonian action of $T = S^1$ and it corresponds to an orthonormal basis of the Lie algebra of $T$. We can define a 1-form $\theta$ by

$$\theta(V) = 1, \ \theta(JV) = 0, \ \theta|_Q = 0,$$

where $\nabla \tau_\ell$ denotes the gradient of $\tau_\ell$ with respect to $g$. By the definition of the moment map, we have $\nabla \tau = JV$. In particular, $\nabla \tau$ is tangent to orbits of the action by $T_C = \mathbb{C}^*$. Of fundamental importance are the following facts (again, taken from [La Nave-Tian09]}
Lemma 2.5. There is a holomorphic frame for which the volume form of $\omega_g$ equals $\det(w)^{-1} \det(h)$.

and:

Lemma 2.6. For any $S^1$-invariant function $\phi \in C^2(M)$, if

$$\gamma = wd\tau - \sqrt{-1} \theta,$$

we have:

\[
\partial f = \partial^h f + \frac{JV(f)}{2} \gamma, \quad \bar{\partial} f = \bar{\partial}^h f + \frac{JV(f)}{2} \bar{\gamma}
\] (9)

and:

\[
\bar{\partial} \bar{\partial} \phi = \sum_{i,j=1}^{n-1} \left( \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} + \frac{1}{4} JV(\phi) \frac{\partial h_{ij}}{\partial \tau} \right) dz_i \wedge d\bar{z}_j - \frac{1}{2} \partial^h (JV(\phi)) \wedge \bar{\gamma} - \frac{1}{2} \gamma \wedge \bar{\partial}^h (JV(\phi)) + \frac{1}{2} JV(JV(\phi)) \gamma \wedge \bar{\gamma}
\] (10)

In particular, taking trace of (10), one gets

\[
\Delta_g f = h^i \left( 4 \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} + JV(f) \frac{\partial h_{ij}}{\partial \tau} \right) + \frac{\partial}{\partial \tau} (JV(f)).
\] (11)

and thus:

\[
w^{-1} \frac{\partial \log \det(h)}{\partial \tau} = \Delta_g \tau - \frac{\partial (w^{-1})}{\partial \tau}
\] (12)

As an immediate corollary we get the following:

Corollary 2.7. Given any $S^1$-invariant functions $f$ and $g$, one has:

\[
\langle \nabla f, \nabla g \rangle_h = \langle \nabla^h f, \nabla^h g \rangle_h
\]

where $\nabla^h$ is the horizontal component of $\nabla$.

Proof. The proof is immediate, since:

\[
\langle \nabla f, \nabla g \rangle_h = h^i \partial_i f \partial_j g + h^i \partial_i g \partial_j f
\]

and since by equation (9) of Lemma 2.6 the horizontal components of $\partial$ and $\partial^h$ are respectively $\partial^h$ and $\bar{\partial}^h$. $\square$

Another important consequence is the following:

Lemma 2.8. Let $\omega_u = \omega_0 + \sqrt{-1} \partial \bar{\partial} u$. Then with respect to the $g_0$ orthogonal decomposition $TM = Q_0 \oplus \langle V \rangle \oplus \langle JV \rangle$ where $Q_0$ is the horizontal distribution of $g_0$. If $Q_u$ is the $g_u$-horizontal distribution, then:

\[
\omega_u := \omega_0 + \sqrt{-1} \partial \bar{\partial} u
\]

\[
= \omega_{h_u} - \frac{\sqrt{-1}}{2} \partial^h (JV(u)) \wedge \bar{\gamma} - \frac{\sqrt{-1}}{2} \gamma \wedge \bar{\partial}^h (JV(u)) + \frac{|V_u|^2}{2} \gamma \wedge \bar{\gamma}
\] (13)
where $\gamma = w_0 d\tau_0 - \sqrt{-1} \theta_0$ and:
\[
\omega_u := \omega_{h_0} + \sqrt{-1} \partial^h \bar{\partial}^h u + \sqrt{-1} \frac{1}{4} \mathcal{J}V(u) \frac{\partial h_0}{\partial \tau}
\]
is the projection of $\omega_u$ onto $\mathcal{O}_0^{(1,1)} \land \mathcal{O}_0^{(1,0)}$ (where $Q_0$ is the horizontal distribution $Q_0$ of $g_0$). In particular, there is a holomorphic frame such that the volume form of $\omega_u$ is:
\[
\det(h_u) |V|^2_u.
\]

**Proof.** The first formula is a straightforward consequence of Lemma 2.6. The assertion about the volume is proved as in [La Nave-Tian09] Lemma 3.3. Alternatively and more conceptually, one can prove the main identity, equation (14), as follows. One knows (cf. [La Nave-Tian09] Corollary 3.5) that the moment map for $\omega + i \partial \bar{\partial} u$ is $\mu_u = \mu_0 + \frac{\mathcal{J}V(u)}{2}$. One also knows (cf. [La Nave-Tian09] Proposition 2.4) that the maps:
\[
\phi_s : M \rightarrow M \text{ and } \phi_{u,s} : M \rightarrow M
\]
defined resp. by $\frac{d \phi}{ds} = \frac{\mathcal{J}V(\phi)}{|V|_u^2}$ and $\frac{d \phi_u}{ds} = \frac{\mathcal{J}V(\phi_u)}{|V|_u^2}$ (and both equal to the identity at $s = 0$) induce biholomorphisms:
\[
\tilde{\phi}_u : X_u(a) := \mu_u^{-1}(a)/S^1 \rightarrow X_u(t + a) := \mu_u^{-1}(a + t)/S^1
\]
and:
\[
\tilde{\phi}_u : X_0(a) := \mu_0^{-1}(a)/S^1 \rightarrow X_0(t + a) := \mu_0^{-1}(a + t)/S^1
\]
Now, fixing a biholomorphism $\pi_a : X_u(a) \rightarrow X_0(a)$ (and one can choose a canonical one by considering the intersections $\mu_0^{-1}(a)$ and $\mu_u^{-1}(a)$ with the integral curves of the flow generated by $\mathcal{J}V$) produces biholomorphisms:
\[
\pi_t : X_u(a + t) \rightarrow X_0(a + t)
\]
by the equation: $\pi_t := \tilde{\phi}_0 \circ \pi_a \circ \tilde{\phi}_u^{-1}$. One thus gets a uniform way of identifying the horizontal distributions $Q_0$ and $Q_u$ by means of a holomorphism. Finally, if $\{z_i\}_{i=1}^m$ are holomorphic coordinates giving rise to a frame for $Q_0$ and $\{\zeta_i\}_{i=1}^m$ the one of $Q_u$ induced by it via $\pi_t$, one has that $\pi_t^* (g_{aij} dz_i \land dz_j) = g_{u\alpha\beta} d\zeta_\alpha \land d\bar{\zeta}_\beta$, where $g_{aij} := g_u(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ and $g_{u\alpha\beta} := g_u(\frac{\partial}{\partial \zeta_\alpha}, \frac{\partial}{\partial \bar{\zeta}_\beta})$. Now we just appeal to the volume formula (cf. Lemma 3.3 in [La Nave-Tian09]):
\[
\det(g_u) = \det(\tilde{h}_u) |V|^2_u
\]
where $\tilde{h}_u = g_{u\alpha\beta} d\zeta_\alpha \land d\bar{\zeta}_\beta$ is the horizontal projection onto $Q_u$. Using the fact that the determinant is invariant under holomorphic transformations, we are done with the second proof as well.

This is an easy consequence of the fact that if $\tau_u$ is the moment map parameter for $\omega_u$, then $\tau_u = \tau_0 + \frac{1}{2} \mathcal{J}V(u)$. In fact this implies that:
\[
d\tau_u = d\tau_0 + \frac{1}{2} \partial^h (\mathcal{J}V(u)) + \frac{1}{2} \bar{\partial}^h (\mathcal{J}V(u)) + \frac{\partial \mathcal{J}V(u)}{\partial \tau} d\tau_0
\]
whence:

\[
d\tau_u \wedge \theta_0 = \left(1 + \frac{\partial JV(u)}{\partial \tau}\right)d\tau_0 \wedge \theta_0 + \frac{1}{2} \delta^h (JV(u)) \wedge \theta_0 + \frac{1}{2} \bar{\delta}^h (JV(u)) \wedge \theta_0
\]

and therefore if we write (using equation (10)):

\[
\omega_u = \left(\omega_{h_0} + \sqrt{-1} \delta^h \bar{\delta}^h u + \sqrt{-1} \frac{1}{4} JV(u) \frac{\partial h}{\partial \tau}\right) - \frac{\sqrt{-1}}{2} \delta^h (JV(u)) \wedge \theta_0
\]

we see that:

\[
\omega_u = \left(\omega_{h_0} + \frac{1}{4} JV(\phi) \frac{\partial h}{\partial \tau}\right) - d\tau_u \wedge \theta_0
\]

Since \(w_u d\tau_u = w_0 d\tau_0\) (being both the dual of the vector field \(JV\)) and since:

\[
d\tau_u \wedge \theta_0 = \frac{1}{w_u} w_u d\tau_u \wedge \theta_0 = -\sqrt{-1} \frac{1}{w_u} \gamma \wedge \bar{\gamma}
\]

we may conclude. \(\square\)

Equation (13), in turn, allows us to prove:

**Lemma 2.9.** In the same hypotheses and with the same notation as above, for any \(S^1\)-invariant function \(f \in C^2(M)\), we have:

\[
\text{tr}_u (\partial \bar{\partial} f) = \text{tr}_u (\partial \bar{\partial} f) + \frac{Z \bar{Z}(f)}{|V|^2_u} \quad (15)
\]

**Proof.** This is a straightforward consequence of Lemma 2.6 and equation (13) in Lemma 2.8. \(\square\)

### 3. The differential operators

Here we intend to introduce (mostly notation) the differential operators that will be involved in the Main differential inequality.

Recall from [La Nave-Tian09] that in symplectic coordinates, if \(g = h_{i\bar{j}} dz_i \otimes \bar{dz}_j + w d\tau^2 + \frac{1}{w} \theta^2\):

\[
\Delta_g f = h^{i\bar{j}} \left(4 \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} + JV(f) \frac{\partial h_{i\bar{j}}}{\partial \tau}\right) + \frac{\partial}{\partial \tau} (JV(f)). \quad (16)
\]

and thus:

\[
w^{-1} \frac{\partial \log \det(h)}{\partial \tau} = \Delta_g \tau - \frac{\partial (w^{-1})}{\partial \tau} \quad (17)
\]

We are interested in the horizontal Laplacian \(\Delta_h f := h^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\). Thanks to equation (16), in invariant form we can write:

\[
\Delta_h f := \Delta_g f - \frac{1}{|V|^2_g} \left(\Delta_g \tau - \frac{JV(|V|^2_g)}{|V|^2_g}\right) JV(f) - \frac{1}{|V|^2_g} JV(JV(f)) \quad (18)
\]
For $S^1$-invariant functions (i.e. $V(f) = 0$) we consider the following second order operator which is closely related to the linearization of the (perturbed) scalar $V$-soliton equation:

$$
\hat{\Delta}^H_{u,\epsilon} f := \Delta_u f - \frac{1}{4(\epsilon + |V|^2_u)} JV(JV(f) = \Delta_u f - \frac{1}{\epsilon + |V|^2_u} Z \bar{Z}(f),
$$

(19)

which is equal to $\Delta^H_u f + \frac{\epsilon}{4|V|^2_u} JV(JV(f)$, where $\Delta^H_u f$ denotes the modified horizontal Laplacian with respect to the metric $g_u$:

$$
\Delta^H_u f := \Delta_h^u f + \frac{1}{\epsilon + |V|^2_u} \left( \Delta_u \tau - \frac{JV(|V|^2_u)}{|V|^2_u} \right) \frac{1}{2} \left( Z(f) + \bar{Z}(f) \right).
$$

We will also make extensive use of the following tensors:

$$
\hat{\hat{g}}^{ij}_{\epsilon, u} := g^{ij}_0 - \frac{Z^i Z^j}{|V|^2_0 + \epsilon}
$$

and:

$$
\hat{\hat{g}}^{ij}_{u, \epsilon} := g^{ij}_u - \frac{Z^i Z^j}{|V|^2_u + \epsilon}
$$

which define (possibly degenerate) Hermitian metrics on $\Omega^{(1, 0)} := T^{(1, 0)}_M$. Observe that one has:

$$
\hat{\Delta}^H_{\epsilon, u} f = \hat{\hat{g}}^{ij}_{\epsilon, u} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}
$$

From now on, we will set:

$$
\hat{g}_{uij} := g_{uij} - \frac{|V|^2_u}{|V|^2_0} \frac{Z_i \bar{Z}_j}{|V|^2_0}
$$

where $Z_i dz_i$ is the holomorphic form such that $g^{ij}_0 Z_i \bar{Z}_j = |V|^2_0$. Note that:

$$
tr_{g^0} \hat{g}_{uij} = \Delta^H u + n - 1
$$

In invariant notation $\hat{g}_{uij}$ induces the tensor:

$$
\hat{g}_u(v, w) := g_u(v, w) - \frac{|V|^2_u}{|V|^2_0} g_0(v, \bar{Z}) g_0(Z, w)
$$

from which the following fact descends trivially:

**Lemma 3.1.** Let $Q \subset TM$ be the distribution of vector fields which are $g_0$-orthogonal to the span of $V$ and $JV$. Then:

$$
\hat{g}_u |_{Q} > 0
$$

and $g_u$ is degenerate in the directions of $V$ and $JV$. Furthermore the signature of $\hat{g}_u$ is $(n - 1, 0)$ and:

$$
tr_{\hat{g}_u |_{Q}} (\hat{g}_u |_{Q}) = \Delta^H_0 u + m
$$
Proof. If \( X \) is in \( Q \), then:

\[
|X|_{g_n}^2 = |X|_u^2 - \frac{g_u(X, \bar{Z}) g_u(Z, X)}{|V|^2_u} = |X|_u^2 > 0
\]

On the other hand, if \( X = aZ \):

\[
|X|_{g_n}^2 = |X|_u^2 - \frac{g_u(X, \bar{Z}) g_u(Z, X)}{|V|^2_u} = a^2 |Z|^2_u - a^2 \frac{|Z|^4_u}{|V|^2_u} = 0
\]

The statement about the signature can be seen by choosing coordinates where \( g_u \) is diagonal, \( Z_i = 0 \) for \( i < n \) and \( g_{0\bar{m}n} = 1 \) at any given point \( p \). Finally, the fact that \( \text{tr}_{g_{0\bar{m}n}} (\hat{g}_u |_{Q}) = \Delta_0^H u + m \) descends from the statement about the signature and elementary properties of tr. \( \square \)

3.1. Basic facts and some notation. In this article we will always work on a Kähler manifold \((M, \omega, J)\). From now on we will denote the complexified Levi–Civita:

\[
\nabla_C : T_C M \otimes C^\infty (T_C M) \rightarrow C^\infty (T_C M)
\]

simply by \( \nabla \). It is a standard fact that on a Kähler manifold:

\[
\nabla_\frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_k} = \Gamma^k_{ij} \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_k} \quad \text{and} \quad \nabla_\frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_k} = \Gamma^k_{ij} \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_k},
\]

independently of the choice of holomorphic coordinates \( z_1, \cdots, z_n \). We will denote:

\[
\nabla^i := \nabla_\frac{\partial}{\partial \bar{z}_i} \quad \text{and} \quad \nabla^\bar{j} := \nabla_\frac{\partial}{\partial z_j}.
\]

It is a standard fact that on a Kähler manifold the \((1,1)\) component of the (Riemannian) Hessian— namely \([\nabla_i \nabla_j]^{(1,1)} = \nabla_i \nabla_j \) —when acting on functions is independent of the metric chosen and it equals the complex Hessian:

\[
\nabla_i \nabla_j f = \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.
\]

We will make ubiquitous use of the following:

**Lemma 3.2.** Let \( Z \) be any holomorphic vector field on a Kähler manifold \((M, g)\). Then, for any \( \epsilon \geq 0 \):

\[
\sqrt{-1} \partial \bar{\partial} \log (|Z|_g^2 + \epsilon) \geq - \frac{\text{Rm}_g(Z, \bar{Z}, \cdots)}{|Z|_g^2 + \epsilon} \quad \text{(20)}
\]

Proof. The two expressions are tensorial, so we can choose special coordinates to show the inequality. By direct calculation, in any normal holomorphic coordinates \( z_1, \cdots, z_n \):

\[
\frac{\partial^2}{\partial z_i \partial \bar{z}_j} (|Z|_g^2) = \frac{\partial^2 g_{kl}}{\partial z_i \partial \bar{z}_j} Z^k \bar{Z}^l + g_{ki} g_{l\bar{j}} - R(g)_{ijkl} Z^k \bar{Z}^l + g_{ki} g_{l\bar{j}} Z^k \bar{Z}^l
\]

Since:

\[
\sqrt{-1} \partial \bar{\partial} \log (|Z|_g^2 + \epsilon) = \sqrt{-1} \frac{\partial \bar{\partial} |Z|_g^2}{|Z|_g^2 + \epsilon} - \sqrt{-1} \frac{\partial |Z|_g^2 + \partial |Z|_g^2}{(|Z|_g^2 + \epsilon)^2} \quad \text{(21)}
\]
in normal holomorphic coordinates:

\[
\sqrt{-1} \partial \bar{\partial} \log(|Z|^2_\mathcal{g} + \epsilon) \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) = - \frac{\text{Rm}_\mathcal{g}(Z, \bar{Z}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})}{(|Z|^2_\mathcal{g} + \epsilon)} + \frac{g_{kl} Z^k_i \bar{Z}^\ell_j - g_{kl} g_{st} Z^k_i \bar{Z}^l_s Z^\ell_j}{(|Z|^2_\mathcal{g} + \epsilon)^2} \tag{22}
\]

On the other hand equation (22) is actually valid in any holomorphic coordinates (as both sides are coordinate independent) and choosing coordinates where \(g_{ij}(p)\) is diagonal and \(Z_k(p) = 0\) for \(k < n\) at any given point \(p\), one finds that:

\[
g_{kl} Z^k_i \bar{Z}^\ell_j \frac{|Z|^2_\mathcal{g} + \epsilon}{|Z|^2_\mathcal{g} + \epsilon} - g_{kl} g_{st} Z^k_i \bar{Z}^l_s Z^\ell_j \frac{|Z|^2_\mathcal{g} + \epsilon}{|Z|^2_\mathcal{g} + \epsilon} = \sum_{k=1}^{n} \sum_{i=1}^{n-1} \frac{|Z|^2_i}{|Z|^2_\mathcal{g} + \epsilon} + \sum_{i,k=1}^{n} \frac{|Z|^2_i}{|Z|^2_n + \epsilon} \geq 0.
\]

whence:

\[
\sqrt{-1} \partial \bar{\partial} \log(|Z|^2_\mathcal{g} + \epsilon) \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) \geq - \frac{\text{Rm}_\mathcal{g}(Z, \bar{Z}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})}{|Z|^2_\mathcal{g} + \epsilon} \tag{23}
\]

as claimed.

□

An immediate corollary of Lemma 3.2 is the following:

**Lemma 3.3.** Let \(Z\) be any holomorphic vector field on a Kähler manifold \((M, g_0)\). Let \(g_\epsilon := g_0 + \sqrt{-1} \partial \bar{\partial} u\) be another Kähler metric in the same class. Then, for any \(\epsilon \geq 0\):

\[
\Delta_0 \log(|Z|^2_u + \epsilon) \geq -g_\epsilon^{ij} \text{Rm}(g_\epsilon)_k \bar{Z}^k \bar{Z}^l \frac{|Z|^2_u}{|Z|^2_u + \epsilon} \tag{24}
\]

**Proof.** This is an immediate consequence of Lemma 3.2 and the fact that in any holomorphic coordinate system: \(\Delta_0 f = g_\epsilon^{ij} f_{ij}\). □

Also we will be using the following symmetries of the Riemannian curvature tensor:

\[
\text{Rm}(x, y, v, w) = -\text{Rm}(x, y, w, v) = -\text{Rm}(y, x, v, w) = \text{Rm}(v, w, x, y)
\]

and the following symmetries of the Riemannian curvature tensor of a Kähler metric:

\[
R_{ijkl} = R_{klij} = R_{jikl}
\]
3.2. Lifting the horizontal curvature. Here we denote $m = n - 1$ where $\text{dim}_\mathbb{C}(M) = n$. We also denote by $Q$ the horizontal distribution of $g$ and write:

$$g = h + |V|^2 g \gamma \wedge \bar{\gamma}$$

where $\gamma := w \, d\tau - \sqrt{-1} \partial \bar{\partial} \theta$.

Also recall that if $g = g_{ij} \omega_i \otimes \omega_j$ then by Cartan’s formulae:

$$\Omega(g)^j_i = -g^{ji} \partial \bar{\partial} g_{i\bar{t}} + g^{sp} g^{jq} \partial g_{ip} \wedge \partial g_{k\bar{q}}$$

(25)

where:

$$\Omega(g)^j_i = \sum_{k,l} R^j_{ikl}(g) dz_k \wedge dz_l$$

Here we derive a formula for the horizontal projection of the Riemannian curvature of $g$. Explicitly:

**Proposition 3.4.** Let $e_1, \ldots, e_m, J e_1, \ldots, J e_m$ be a unitary frame for $h$. Then one has:

$$R_{ijkl}(g) = R_{ijkl}(h) - h^{\bar{m}} h^s \frac{1}{2} JV(h_{skl}) \frac{\partial h_{\bar{m}l}}{\partial \tau}$$

Proof. According to Lemma 2.6 and to formula (25), if we denote by $[\Omega]^h$ the projection of a 2-form onto $Q^{(0,1)} \wedge Q^{(1,0)}$ one has:

$$[\Omega(g)^j_i]^h = -g^{ji} \left( \partial h \partial h g_{i\bar{t}} + \frac{1}{4} JV(g_{i\bar{t}}) \frac{\partial h}{\partial \tau} \right)$$

$$+ g^{sp} g^{jq} \left( \partial h g_{ip} \right) \wedge \left( \partial h g_{k\bar{q}} \right)$$

and the claim follows, using that $g = h + w d\tau^2 + \frac{1}{w} \theta^2$ (and the fact that for any $i$, $g(e_i, v) = 0$ unless $v \in Q$). \qed

4. Non existence of Steady and Expanding Compact $V$-solitons

Here we intend to prove that on a compact manifold, there is no solution to the $\epsilon$-perturbed $V$-soliton equation:

$$\text{Ric}(g_u) + \sqrt{-1} \partial \bar{\partial} \log(\epsilon + |V|^2) = \lambda g_u$$

(26)

unless $\lambda > 0$. This is a generalization of a result of Perelman’s on gradient solitons. In analogy with solitons, we set:

**Definition 4.1.** A solution to equation (26) is called an expanding, steady or shrinking $V$-soliton if, respectively, $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

**Theorem 4.2.** If $g_u$ is a solution to equation (26) one has:

$$\hat{\Delta}_{u,\epsilon} |V|^2_u \geq -\lambda |V|^2_u$$

(27)

In particular, on a closed compact manifold $M$ (or any time the minimum principle is applicable), for any $\epsilon > 0$ there is no nontrivial solution to the to the $\epsilon$-perturbed $V$-soliton equation:

$$\text{Ric}(g_u) + \sqrt{-1} \partial \bar{\partial} \log(\epsilon + |V|^2) = \lambda g_u$$
unless \( \lambda > 0 \). Here we refer to a solution to the Einstein equation as a trivial \( V \)-soliton.

**Proof.** We calculate:

\[
\hat{\Delta}_{u,\epsilon}|V|^2_u = -\hat{g}^{ki}_{u,\epsilon} R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j + |\nabla Z|^2_{g_u,\epsilon}
\]  
(28)

where \( |\nabla Z|^2_{g_u,\epsilon} = \hat{g}^{ki}_{u,\epsilon} g_{ui\bar{j}} Z^i \bar{Z}^j \) and since (by definition of \( \hat{g}^{ki}_{u,\epsilon} \)):

\[
\hat{g}^{ki}_{u,\epsilon} R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j = g^{ki}_u R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j - \frac{Z^k \bar{Z}^i}{|V|^2_u + \epsilon} R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j
\]

\[
= \text{Ric}(g_u)(Z, \bar{Z}) - \frac{Z^k \bar{Z}^i}{|V|^2_u + \epsilon} R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j
\]

and therefore, using the \( V \)-soliton equation (26) and Lemma 3.2, we get:

\[
\hat{g}^{ki}_{u,\epsilon} R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j = -\sqrt{-1} \partial_i \bar{\partial}_j \log(\epsilon + |V|^2_u) Z^i \bar{Z}^j - \frac{Z^k \bar{Z}^i}{|V|^2_u + \epsilon} R_{(g_u)i\bar{j}k\bar{l}} Z^i \bar{Z}^j + \lambda |V|^2_u \leq \lambda |V|^2_u
\]

Therefore, coupling this with eq. (28) yields:

\[
\hat{\Delta}_{u,\epsilon}|V|^2_u \geq -\lambda |V|^2_u
\]

and if \( \lambda \leq 0 \) the minimum principle implies that \( |V|^2_u \) is constant. \( \square \)

### 5. The Main Differential Inequalities

Recall that in equation (18) we defined:

\[
\Delta_{h} f := \Delta_{g} f - \frac{1}{|V|^2_g} \left( \Delta_{g} \tau - \frac{JV(V^2_g)}{|V|^2_g} \right) J V(f) - \frac{1}{|V|^2_g} J V(J V(f))
\]

Observe that if \( h \) denotes the projection of \( g \) on \( Q^{(1,0)} \land Q^{(0,1)} \) (here \( Q \) is the \( g \)-horizontal distribution), then:

\[
\Delta_{h} f = \text{tr}_h(\partial^h \bar{\partial}^h u)
\]

where \( \partial^h \bar{\partial}^h \) is the horizontal complex Hessian and

\[
\Delta_{\bar{h}} f := \left( g^{i\bar{j}} - \frac{Z^i \bar{Z}^j}{|V|^2_g} \right) f_{i\bar{j}} = \text{tr}_h(\partial \bar{\partial} u)
\]

We will start by proving the following generalization of a standard fact:

**Lemma 5.1.** Let \((M, g_0)\) be any Kähler metric and let \( u \in C^4(M) \) such that \( g_0 + \partial \bar{\partial} u > 0 \). Then one has the following identities:

\[
\Delta_0 (\Delta_0 u) = g^{\bar{i}\bar{j}} g_0^{i\bar{k}} u_{i\bar{j}\bar{k}} + g^{\bar{i}\bar{j}} R(g_0)_{i\bar{k}j\bar{l}} g_{ui\bar{j}} - g^{\bar{i}\bar{j}} R(g_0)_{k\bar{i}}
\]
and:
\[ \Delta u (\Delta^H_0 u) = h^{kl}_{ij} g_{ik} g_{lj} u_{ijkl} + h^{kl}_{ij} g_{ik} \tilde{g}_{lj} R_{ijkl}(g_0) \tilde{g}_{ijkl} \]
\[ - h^{kl}_{ij} g_{ik} R_{ijkl}(g_0) + \frac{ZZ(\Delta^H_0 u)}{|V|^2} \]
where the indices in \( u_{ijkl} \) indicate differentiation with respect to holomorphic coordinates of \( M \).

**Proof.** The first identity follows easily from considering normal holomorphic coordinates for \( g_0 \) and writing \( \Delta u (\Delta_0 u) = g^{kl}_{ik} \left( g^{ij}_{ik} u_{ijkl} \right)_{kl} \).

The main difficulty in computing the Hessian of \( \Delta_h u \) stems from the difference between the complex Hessian \( \partial \bar{\partial} \) and its horizontal (with respect to \( g_0 \)) counterpart \( \partial^h \bar{\partial}^h \) (cf. Lemma 2.6). In fact
\[ \Delta^H_0 u = \text{tr}_h(\partial \bar{\partial} u) = \text{tr}_h \left( \partial^h \bar{\partial}^h u + \frac{1}{4} JV(u) \frac{\partial h_0}{\partial \tau} \right) \]
Thus, using equation (15) in Lemma 2.9
\[ \text{tr}_u \left( \partial \bar{\partial}(\Delta^H_0 u) \right) = \text{tr}_u \left( \partial \bar{\partial}(\Delta_0 u) \right) + \frac{Z \bar{Z}(\Delta^H_0 u)}{|V|^2} \]  
(29)
We now analyze the term \( \text{tr}_u \left( \partial \bar{\partial}(\Delta^H_0 u) \right) \). Remark that we only need to understand the horizontal projection of \( \partial \bar{\partial}(\Delta_h u) \) onto \( Q^{(1,0)} \wedge Q^{(0,1)}_0 \) (since we are taking the trace with respect to the horizontal projection of \( \omega_u \)). Given a point \( p \) on which \( V(p) \neq 0 \), choose a horizontal holomorphic frame for \( Q^{(1,0)}_0 : \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m} \) centered at \( p \) such that:
\[ \frac{\partial}{\partial z_i} h_{0kl}(p) = \frac{\partial}{\partial z_j} h_{0kl}(p) = 0 \]
and:
\[ \frac{\partial^2}{\partial z_i \partial z_k} h^{kl}_{ij} = R(h_0)^{kl}_{ij} \]
Next observe that:
\[ \text{tr}_u \left( \partial \bar{\partial}(\Delta^H_0 u + m) \right) = h^{kl}_{ij} \partial \bar{\partial} \Delta^H_0 u + h^{kl}_{ij} \partial \bar{\partial} m \]
\[ = h^{kl}_{ij} \partial \bar{\partial} \Delta^H_0 u + h^{kl}_{ij} \partial \bar{\partial} \Delta_0 u + \frac{1}{4} JV(h_0) \frac{\partial h_{0kl}}{\partial \tau} \]
(30)
Now, observe that by Corollary 2.7:
\[ h^{kl}_{ij} \partial \bar{\partial} \Delta_0 u + h^{kl}_{ij} \partial \bar{\partial} \Delta_0 u = h^{kl}_{ij} \partial \bar{\partial} \Delta_0 u + h^{kl}_{ij} \partial \bar{\partial} \Delta_0 u = 0 \]
since for our choice of coordinates \( \nabla_k^h(h_{ij}) = \nabla_k^h(h_{ij}) = 0 \). Also, using Lemma 2.6 again:
\[ \nabla_k \nabla_l(h_{ij}^h) = \partial_k \bar{\partial}_l(h_{ij}^h) = \partial_k \bar{\partial}_l(h_{ij}^h) + \frac{1}{4} JV(h_0) \frac{\partial h_{0kl}}{\partial \tau} \]
(31)
therefore, in our choice of coordinates:
\[ \nabla_k \nabla_l(h_{ij}^h) = R_{ijkl}(h) - h^{ji} h^{si} \frac{JV(h_{si})}{4} \frac{\partial h_{kl}}{\partial \tau} \]
Recall now that by Proposition 3.4, in any frame:
\[ R_{ijkl}(g) = R_{ijkl}(h) - h^{li}h^{kj}J V(h_{li}) \frac{\partial h_{ki}}{\partial \tau} \]
which, in conjunction with formulae (30) and (31), yields:
\[ \nabla_k \tilde{\nabla}_l (h_{ij}) = R_{ijkl}(g) \tag{32} \]
Collecting what we have shown so far, we can infer:
\[ \Delta_u (\Delta^H_0 u) = h^{kl}_{ki} h^{kj}_{k0} R_s(ki) \partial \tilde{\partial} u \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) + \frac{Z \tilde{Z} (\Delta^H_0 u)}{|V|^2} \]
The statement is now proven, since:
\[ h^{kl}_{ki} h^{kj}_{k0} R_s(ki) \partial \tilde{\partial} u \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) = h^{kl}_{ki} h^{kj}_{k0} R_s(ki) \frac{\partial \tilde{\partial} u \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)}{\partial \tau} + h_{0ij} \]
\[ \square \]

5.1. The horizontal differential inequality. Recall that:
\[ \tilde{g}_{ij} := g_{ij} - \frac{|V|^2}{|V|^2} Z_i Z_j \]
where \( Z_i dz_i \) is the holomorphic form such that \( g_0 Z_i Z_j = |V|^2 \).
Recall that \( \tilde{g}_{ij} := \tilde{g}_{ij} - \frac{Z_i Z_j}{|V|^2}, \tilde{g}_{ij} := g_{ij} - \frac{Z_i Z_j}{|V|^2} \) and \( \tilde{\Delta}_{\epsilon, u} f := \tilde{g}_{ij} f_{ij} \).
In the sequel we will need:

Lemma 5.2. Let \( g_0 \) be a given Kähler metric and let \( g_u := g_0 + \sqrt{-1} \partial \tilde{\partial} u \) such that \( u \in C^4(M) \) is a solution of equation (8). Then one has:
\[ \Delta^H \log(|Z|^2 + \epsilon) \geq -\tilde{g}_{ij} g_{ij} \partial \tilde{\partial} u \tilde{g}_{ij} g_{ij} |Z|^2 + \epsilon - \tilde{g}_{ij} R(g_0) \frac{Z^k \tilde{Z}^l}{|Z|^2 + \epsilon} + \tilde{g}_{ij} \partial_k \partial_{\tilde{\partial} u} \frac{Z^k \tilde{Z}^l}{|Z|^2 + \epsilon} \tag{33} \]

Proof. First recall the standard relation:
\[ R(g_u)_{ijkl} = g_{pl} \partial_i g_{ukq} \tilde{g}_{lj} g_{apil} + R(g_0)_{ijkl} - \partial_k \partial_{\tilde{\partial} u} \tilde{g}_{ij} \]
which readily follows from the formula of the Riemannian curvature \( R(g_u)_{ijkl} = -\frac{\partial^2 g_{ui}^j}{\partial z_k \partial z_l} + g^p_{ij} \partial_l g_{uqp} \partial_j g_{apil} \) and the fact that in normal holomorphic coordinates for \( g_0: \frac{\partial^2 g_{ui}^j}{\partial z_k \partial z_l} = R(g_0)_{ijkl} - \partial_k \partial_{\tilde{\partial} u} \tilde{g}_{ij} \).
Recall from eq. (24) of Lemma 3.3, that one has:
\[ \Delta^H \log(|Z|^2 + \epsilon) \geq -\tilde{g}_{ij} R(g_u)_{ij} \frac{Z^k \tilde{Z}^l}{|Z|^2 + \epsilon} \]
whence, upon using the standard relation above (equation (34)), one finds:

\[
\Delta_0^H \log (|Z|_g^2 + \epsilon) \geq -\hat{g}^{ij}_0 g_u^{pq} \partial_i g_{u k q} \bar{\partial}_j g_{u p l} \frac{Z^k \bar{Z}^l}{|Z|_g^2 + \epsilon} - \hat{g}^{ij}_0 R(g_0)_{ij kl} \frac{Z^k \bar{Z}^l}{|Z|_g^2 + \epsilon}
\]

\[+ \hat{g}^{ij}_0 \partial_k \partial_l u_{ij} \frac{Z^k \bar{Z}^l}{|Z|_g^2 + \epsilon}.
\]

(35)

\[\] □

From now on we will use the following operator:

\[
\Delta_{\beta, u} f := \hat{\Delta}_u f + \beta Z \bar{Z} \left( f_{ij} \right) = \left( h^{ij}_u + \beta Z^j \bar{Z}^i \right) f_{ij}.
\]

In the application we will take:

\[
\beta = \frac{1}{|V|^2 - u} - \frac{1}{|V|^2} = \frac{\epsilon}{|V|^2} \frac{|V|^2}{|V|^2 + \epsilon}.
\]

so that:

\[
\Delta_{\beta, u} f = \hat{\Delta}_{u, \epsilon} f.
\]

We are now ready to prove:

**Proposition 5.3.** Let \( g_0 \) be a given Kähler metric and let \( g_u := g_0 + \sqrt{-1} \partial \bar{\partial} u \) such that \( u \in C^4(M) \) is a solution of equation (8). Then one has:

\[
\hat{\Delta}_{\epsilon, u} \left( \Delta_0^H u \right) \geq \hat{g}^{ij}_0 g_u^{pq} \partial_i g_{u k q} \bar{\partial}_j g_{u p l} + \hat{g}^{k \bar{l}}_0 g_u^{ij} g_{u k q} \bar{\partial}_j g_{u p l} + \hat{g}^{ij}_0 R(g_0)_{ij kl} - \partial_k \partial_l u_{ij}.
\]

(36)

\[\]

**Proof.** From now on, we will denote \( \nabla_{u, \partial g_u} \) by \( \nabla'_k \) and \( \nabla_{u, \partial g_u} \) by \( \nabla'_{k} \). Taking the complex Hessian of the expression \( g_{u ij} = g_{0 u ij} + u_{ij} \) one gets:

\[
R(g_u)_{ij kl} = g_u^{pq} \partial_i g_{u k q} \bar{\partial}_j g_{u p l} + R(g_0)_{ij kl} - \partial_k \partial_l u_{ij}.
\]

(37)

Contracting with \( g^{ij}_0 \) and with \( \hat{g}^{ij}_0 \) then yields:

\[
\hat{g}^{ij}_0 R(g_u)_{ij} = \hat{g}^{ij}_0 g^{ij}_0 g_u^{pq} \partial_i g_{u k q} \bar{\partial}_j g_{u p l} + \hat{g}^{ij}_0 g_u^{ij} g_{u k q} \bar{\partial}_j g_{u p l} + \hat{g}^{ij}_0 g_u^{ij} R(g_0)_{ij kl} - \hat{g}^{ij}_0 g^{ij}_0 \partial_k \partial_l u_{ij}.
\]

(38)

We next observe that from Lemma 5.1 one knows:

\[
\Delta_u \left( \Delta_0^H u \right) = h^{k \bar{l}}_0 \hat{g}^{ij}_0 \partial_i u_{j k} + h^{k \bar{l}}_0 \hat{g}^{ij}_0 \partial_i u_{j k} + R_{ij kl}(g_0) \hat{g}^{ij}_0
\]

\[= h^{k \bar{l}}_0 \hat{g}^{ij}_0 R_{ij kl}(g_0) + \frac{Z \bar{Z} (\Delta_0^H u)}{|V|^2 - u}.
\]

and therefore that:

\[
-h^{k \bar{l}}_0 \hat{g}^{ij}_0 \partial_i u_{j k} + h^{k \bar{l}}_0 \hat{g}^{ij}_0 R_{ij kl}(g_0) = -\Delta_u \left( \Delta_0^H u \right) + h^{k \bar{l}}_0 \hat{g}^{ij}_0 \partial_i u_{j k} + \frac{Z \bar{Z} (\Delta_0^H u)}{|V|^2 - u}.
\]

which yields, after observing that \( h^{k \bar{l}}_0 = g^{k \bar{l}}_u - \frac{g^{k \bar{l}}_u Z \bar{Z}}{|V|^2 - u} \):
\[ g^{k\ell}_{u} g^{\ell j}_{0} \left( -u_{ij\ell k} + R_{ij\ell k}(g_0) \right) = -\frac{Z^{k \bar{l}}}{|V|^{2}_{u}} \tilde{g}^{\ell j}_{0} \left( -u_{ij\ell k} + R_{ij\ell k}(g_0) \right) + \hat{\Delta}_{u} (\Delta^{H}_{0} u) \]

\[-\hat{g}_{u} g^{k\ell}_{0} \hat{g}^{\ell j}_{0} R_{sikl}(g_0) \tilde{g}_{uij}\]

where:

\[ \hat{\Delta}_{u} f = \Delta_{u} - \frac{Z^{k \bar{l}}}{|V|^{2}_{u}} f_{kl} = h^{k\ell}_{u} f_{kl} \]

and therefore equation (38) can be rewritten as:

\[ g^{k\ell}_{0} R(g_u)_{ij} = g^{k\ell}_{0} g^{lq}_{u} \partial_{i} g_{ukq} \partial_{j} g_{up\bar{q}} - \hat{\Delta}_{u} (\Delta^{H}_{0} u) + h^{k\ell}_{u} g^{lq}_{0} g^{s j}_{0} R_{sikl}(g_0) \tilde{g}_{uij} \]

\[ + \frac{Z^{k \bar{l}}}{|V|^{2}_{u}} \hat{g}^{\ell j}_{0} \left( -u_{ij\ell k} + R_{ij\ell k}(g_0) \right). \]

We now make use of the \((\epsilon\text{-perturbed})\) V-soliton equation:

\[ R(g_u)_{ij} = R(-\epsilon \log(|V|_{u}^{2} + \epsilon)) \]

to deduce:

\[ \hat{\Delta}_{u} (\Delta^{H}_{0} u) = g^{k\ell}_{0} g^{lq}_{u} \partial_{i} g_{ukq} \partial_{j} g_{up\bar{q}} + h^{k\ell}_{u} g^{lq}_{0} g^{s j}_{0} R_{sikl}(g_0) h_{uij} + \Delta^{H}_{0} F \]

\[ + \frac{Z^{k \bar{l}}}{|V|^{2}_{u}} \hat{g}^{\ell j}_{0} \left( -u_{ij\ell k} + R_{ij\ell k}(g_0) \right). \]

We now appeal to equation (33) to deduce that:

\[ \hat{\Delta}_{u} (\Delta^{H}_{0} u) \geq g^{k\ell}_{0} g^{lq}_{u} \partial_{i} g_{ukq} \partial_{j} g_{up\bar{q}} + h^{k\ell}_{u} g^{lq}_{0} g^{s j}_{0} R_{sikl}(g_0) h_{uij} + \Delta^{H}_{0} F \]

\[ + \frac{Z^{k \bar{l}}}{|V|^{2}_{u}} \hat{g}^{\ell j}_{0} \left( -u_{ij\ell k} + R_{ij\ell k}(g_0) \right) \]

\[ - \hat{g}^{k\ell}_{0} g^{lq}_{u} \partial_{i} g_{ukq} \partial_{j} g_{up\bar{q}} \frac{Z^{k \bar{l}}}{|Z|^{2}_{g} + \epsilon} - \hat{g}^{\ell j}_{0} R(g_0)_{ij kl} \frac{Z^{k \bar{l}}}{|Z|^{2}_{g} + \epsilon} \]

\[ + \hat{g}^{k\ell}_{0} \partial_{i} \partial_{u} \tilde{g}^{l j}_{0} \frac{Z^{k \bar{l}}}{|Z|^{2}_{g} + \epsilon}. \]

\[ = g^{k\ell}_{0} g^{lq}_{u} \partial_{i} g_{ukq} \partial_{j} g_{up\bar{q}} + h^{k\ell}_{u} g^{lq}_{0} g^{s j}_{0} R_{sikl}(g_0) h_{uij} \]

\[ + \Delta^{H}_{0} F + \beta Z^{k \bar{l}} \hat{g}^{\ell j}_{0} \left( -u_{ij\ell k} + R_{ij\ell k}(g_0) \right) \]

We next observe that:

\[ \hat{\Delta}_{u} (\Delta^{H}_{0} u + m) = \hat{\Delta}_{u} (\Delta^{H}_{0} u + m) + \beta Z \tilde{Z}(\Delta^{H}_{0} u + m) \]

and that (since \(\Delta^{H}_{0} u = h^{k\ell}_{0} u_{ij}\)):

\[ Z \tilde{Z}(\Delta^{H}_{0} u + m) = Z^{k \bar{l}} \nabla_{k} \nabla_{l}(h^{k\ell}_{0}) u_{ij} + Z(h^{k\ell}_{0}) \tilde{Z}(u_{ij}) + Z(u_{ij}) \tilde{Z}(h^{k\ell}_{0}) + h^{k\ell}_{0} Z^{k \bar{l}} u_{ij kl} \]

\[ = Z^{k \bar{l}} h^{k\ell}_{0} R_{sikl}(g_0) u_{ij} + Z(h^{k\ell}_{0}) \tilde{Z}(u_{ij}) + Z(u_{ij}) \tilde{Z}(h^{k\ell}_{0}) + h^{k\ell}_{0} Z^{k \bar{l}} u_{ij kl} \]
where we have used formula (32) which says \( \nabla_k \nabla_l (h_0^{ij}) = h_0^{ij} \partial_k \partial_l R_{stkl}(g_0) \), in coordinates as therein, for which in particular: \( \nabla_k (h_0^{ij}) = \nabla^k (h_0^{ij}) + Z(h_0^{ij}) \) and \( \nabla_l (h_0^{ij}) = \nabla^l (h_0^{ij}) + Z(h_0^{ij}) \) (since \( \nabla_k^k (h_0^{ij}) = \nabla^l (h_0^{ij}) = 0 \)).

Whence, going back to equation (40):

\[
\Delta_{\nu,u} \left( \Delta^H_{0} u \right) \geq h_0^{ij} \partial_{u} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \left( h_0^{ij} + \beta Z^k \bar{Z}^l \right) g_0^{ij} \partial_{u} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \Delta^H_{0} F + \beta Z^k \bar{Z}^l g_0^{ij} \left( \partial_{u} \partial_{ij} + R_{uijkl}(g_0) \right) + 2 \beta J V(h_0^{ij}) J V(u_{ij}) \]

(41)

after having used that:

\[
\beta Z^k \bar{Z}^l h_0^{ij} \bar{h}_0^{kl} R_{stkl}(g_0) u_{ij} = \beta Z^k \bar{Z}^l h_0^{ij} \bar{h}_0^{kl} R_{stkl}(g_0) u_{ij} - \beta Z^k \bar{Z}^l h_0^{ij} \bar{h}_0^{kl} R_{stkl}(g_0) u_{ij}
\]

and that:

\[
\beta \left( Z(h_0^{ij}) Z(u_{ij}) + Z(u_{ij}) Z(h_0^{ij}) \right) = 2 J V(h_0^{ij}) J V(u_{ij})
\]

Next we observe that (recalling that \( \partial_{u} g_{\nu}^{pq} = g_{\nu}^{pq} - \frac{Z^k \partial_{u} g_{\nu}^{kl}}{|V|^2_{u}} \)):

\[
\partial_{u} g_{\nu}^{pq} \partial_{u} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} = \partial_{u} g_{\nu}^{pq} \partial_{u} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \partial_{u} g_{\nu}^{pq} \partial_{u} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \frac{Z^p Z^q}{|V|^2_{u}} \partial_{u} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \partial_{u} g_{\nu}^{pq} \partial_{u} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \frac{Z^p Z^q}{|V|^2_{u}} \partial_{u} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} + \beta Z^k \bar{Z}^l \frac{Z^p Z^q}{|V|^2_{u}} \partial_{u} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \]

(42)

having used in the last equality that:

\[
\partial_{u} g_{\nu}^{kl} = \partial_{u} g_{\nu}^{kl} + \beta Z^k \bar{Z}^l.
\]

We now observe that inequality (36) follows from equations (41) and (42) along with the following two facts:

(1) one has that:

\[
\partial_{u} g_{\nu}^{kl} Z^p Z^q \partial_{u} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \geq 0
\]

and in particular:

\[
\partial_{u} g_{\nu}^{kl} g_{s,u} \partial_{s} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \geq \beta \partial_{u} g_{\nu}^{kl} Z^k \bar{Z}^l \frac{Z^p Z^q}{|V|^2_{u}} \partial_{u} \partial_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u} \partial_{p} \partial_{q} g_{u}
\]
(2) one has:
\[
Z(h_0^{ij}) \bar{Z}(u_{ij}) + Z(u_{ij}) \bar{Z}(h_0^{ij}) + \beta g_0^{ij} Z^k \bar{Z}^l \frac{Z^p \bar{Z}^q}{|V|^2_u} \partial_i g_{uqk} \bar{\partial}_j g_{upl}
\]
\[
\geq -\beta \left| V \right|^2_u \frac{\left| \nabla Z \right|^2_{h_0}}{|V|^2_0}
\]
and in particular, since \( \beta |V|^2_u = \frac{\epsilon}{|V|^2_0 + \epsilon} \leq 1 \):
\[
Z(h_0^{ij}) \bar{Z}(u_{ij}) + Z(u_{ij}) \bar{Z}(h_0^{ij}) + \beta g_0^{ij} Z^k \bar{Z}^l \frac{Z^p \bar{Z}^q}{|V|^2_u} \partial_i g_{uqk} \bar{\partial}_j g_{upl}
\]
\[
\geq -\frac{|\nabla Z|^2_{h_0}}{|V|^2_0}
\]
The proof of item (1) is an easy consequence of the fact that in holomorphic coordinates \( \{z_1, \ldots, z_n\} \) centered at some point \( p \) diagonalizing \( g_0^{ij} \) and holomorphic coordinates \( \{w_1, \ldots, w_n\} \) (still centered at \( p \)) diagonalizing \( g_0^{\alpha \beta} = \check{g}_u (dw_\alpha, dw_\beta) = \lambda_\alpha \delta_{\alpha \beta} \) and such that the \( Z = Z^n \frac{\partial}{\partial w_n} \) (at any given point such that \( V(p) \neq 0 \)) then one has:
\[
\check{g}_0^{ij} g_u \check{g}_u^{\alpha \beta} \frac{Z^n \bar{Z}^\alpha \bar{Z}^\beta}{|V|^2_u} \partial_i g_{u\alpha \beta} \partial_j g_{u\beta \alpha} = \sum_{i, \alpha = 1}^{n-1} \lambda_\alpha \frac{|Z^{\alpha}|^2}{|Z^n|^2} \partial_i (u_{\alpha n}) \bar{\partial}_j (u_{\beta n}) > 0
\]
On the other hand, the proof of item (2) follows from the following two observations. First we remark that since \( h_0^{ij} = g_0^{ij} - \frac{Z^i \bar{Z}^j}{|V|^2_0} \):
\[
\beta \left( Z(h_0^{ij}) \bar{Z}(u_{ij}) + Z(u_{ij}) \bar{Z}(h_0^{ij}) \right) = -\beta \left( Z \left( \frac{Z^i \bar{Z}^j}{|V|^2_0} \right) \bar{Z}(u_{ij}) + \bar{Z} \left( \frac{Z^i \bar{Z}^j}{|V|^2_0} \right) Z(h_0^{ij}) \right)
\]
In coordinates where \( g_0 \) is diagonal and \( Z^k(p) = 0 \) for \( k < n \), one has:
\[
-\beta \left( Z \left( \frac{Z^i \bar{Z}^j}{|V|^2_0} \right) \bar{Z}(u_{ij}) + \bar{Z} \left( \frac{Z^i \bar{Z}^j}{|V|^2_0} \right) Z(h_0^{ij}) \right) = -\beta \left( \sum_{i=1}^{n-1} Z^i_n \bar{Z}^i_n u_{in} + \sum_{i=1}^{n-1} Z^n_n Z^i_n u_{in} \right)
\]
and using Cauchy-Schwarz:
\[
\sum_{i=1}^{n-1} \left( Z^i_n \bar{Z}^n_n u_{in} + Z^n_n \bar{Z}^i_n u_{in} \right)
\]
\[
= \sum_{i=1}^{n-1} \left( \left( Z^i_n \sqrt{1 + u_{in}} \right) \bar{Z}^n_n \sqrt{1 + u_{in}} u_{in} + \left( \bar{Z}^i_n \sqrt{1 + u_{in}} \right) Z^n_n \sqrt{1 + u_{in}} u_{in} \right)
\]
\[
\leq \sum_{i=1}^{n-1} \left( |Z^i_n|^2 (1 + u_{in}) + |Z^n_n|^2 u_{in} u_{in} \right) = \sum_{i=1}^{n-1} \left( \frac{|Z^i_n|^2}{|V|^4_u} |V|^2_u + \frac{|Z^n_n|^2}{|V|^4_u} |Z|^2 u_{in} u_{in} \right)
\]
\[
= \frac{|\nabla Z|^2_{h_0}}{|V|^2_u} |V|^2_u + g_0^{ij} Z^k \bar{Z}^l \frac{Z^p \bar{Z}^q}{|V|^2_u} \partial_i g_{ukq} \bar{\partial}_j g_{upl}
\]
which concludes the proof. \( \square \)
In order to take care of the bad terms in the differential inequality in eq. (36), we now have two standard possibilities, i.e., we either consider the quantity \( \hat{\Delta}_{v,u} (\log (\Delta_0 u + m) + Cu) \) or the quantity \( \hat{\Delta}_{v,u} (e^{-au}(\Delta_0 u + m)) \) \(^3\); we will choose the latter expression. We first prove the following elementary fact:

**Lemma 5.4.** Given any \( u \in C^{4,\alpha}(M) \) and \( a \in \mathbb{R} \), one has:

\[
\hat{\Delta}_{u,e} \left( e^{-au}(\Delta_0^H u + m) \right) \geq e^{-au} \hat{\Delta}_{u,e}(\Delta_0^H u + m) - ae^{-au}(\Delta_0^H u + m) \hat{\Delta}_{u,e} u - \frac{e^{-au}}{\Delta_0^H u + m} \left| \nabla_{\hat{g}_{u,e}} (\Delta_0^H u) \right|_{{h_u}}^2 + 2\beta JV(e^{-au})JV(\Delta_0^H u + m)
\]

where \( |\nabla_{\hat{g}_{u,e}} f|_{h_u}^2 = h_{ij}^u f_i f_j \)

**Proof.** First off, one trivially has:

\[
\hat{\Delta}_{u,e} \left( e^{-au}(\Delta_0^H u + m) \right) = e^{-au} \hat{\Delta}_{u,e}(\Delta_0^H u + m) + \hat{g}_{u,e}^{ij} \left( (\Delta_0^H u + m)_i (e^{-au})_j + (\Delta_0^H u + m)_j (e^{-au})_i \right) + (\Delta_0^H u + m) \left( -ae^{-au} \hat{\Delta}_{u,e} u + a^2 e^{-au}\hat{g}_{u,e}^{ij} u_i u_j \right)
\]

On the other hand, by Lemma 2.6, for any \( S^1 \)-invariant function \( \phi \in C^2(M) \), if \( \gamma = w d\tau - \sqrt{-1}\theta \), we have:

\[
\partial f = \partial h f + \frac{JV(f)}{2} \gamma, \quad \bar{\partial} f = \bar{\partial} h f + \frac{JV(f)}{2} \bar{\gamma}
\]

thus:

\[
\hat{g}_{u,e}^{ij} \left( (\Delta_0^H u + m)_i (e^{-au})_j + (\Delta_0^H u + m)_j (e^{-au})_i \right) = \hat{h}_{u}^{ij} \left[ (\Delta_0^H u + m)_i (e^{-au})_j + (\Delta_0^H u + m)_j (e^{-au})_i \right] + 2\beta JV(e^{-au})JV(\Delta_0^H u + m)
\]

Next observe that by Cauchy-Schwartz one has:

\[
h_{u}^{ij} (a(\Delta_0^H u)_i u_j + a(\Delta_0^H u)_j u_i) \leq h_{u}^{ij} \left( a^2 u_i u_j (\Delta_0^H u + m) + \frac{(\Delta_0^H u)_i (\Delta_0^H u)_j}{(\Delta_0^H u + m)} \right)
\]

\[\square\]

We will also need the following:

**Lemma 5.5.** Let \( \hat{g}_{u}^{ij} := g_{u}^{ij} - \frac{Z^2 g_{ij}}{|Z|^2} \). Then:

\[
\hat{g}_{u}^{ij} g_{0ij} \geq \left( \frac{|V|^2 |V_0|^2}{(|V|^2 + \epsilon) \det(g_0)} \right)^{\frac{1}{m-1}} e^{-\frac{\beta}{m-1} (m + \Delta_0^H u)} \]

\[\text{\footnote{Here } m := n - 1, \text{ and it'll be clear from what follows why one takes this particular number in the given expression.}}\]
Proof. At any given point $p$ choose holomorphic coordinates such that $g_{i\bar{j}}(p) = \lambda_i \delta_{i\bar{j}}$ and $Z^k(p) = 0$ for $k \neq n$ and $g_{0\bar{i}} = 1$. Then $\hat{g}_{i\bar{j}}(p) = \hat{\lambda}_i \delta_{i\bar{j}}$ where $\hat{\lambda}_i = \frac{1}{\lambda_i}$ if $i = 1, \cdots, n-1$ and $\hat{\lambda}_n = 0$. Thus:

$$\hat{g}_{i\bar{j}} g_{0\bar{i}} = \sum_{i=1}^{n-1} \hat{\lambda}_i ^{n-1}$$

and using the arithmetic/geometric mean inequality (and that $m := n - 1$):

$$\hat{g}_{i\bar{j}} g_{0\bar{i}} = \sum_{i=1}^{n-1} \hat{\lambda}_i ^{n-1} \geq \left( \frac{\sum_{i=1}^{m} (\hat{\lambda}_i ^{n-1})^{n-1}}{\prod_{i=1}^{m} (\hat{\lambda}_i ^{n-1})^{n-1}} \right) ^{\frac{1}{n-1}} = \left( \frac{|V|^2 |V_0|^2}{(|V|^2 + \epsilon) \det(g_0)} \right) ^{\frac{1}{n-1}} e^{-\frac{mF}{m-1}} (m + \Delta_0^H u)^{\frac{1}{m-1}}$$

having used that at the given point $\frac{1}{\hat{\lambda}_i} = \lambda_i$ for $i = 1, \cdots, m = n - 1$ and that

$$\lambda_n = \frac{n^2 |\bar{Z}|^2}{|Z|^2} = \frac{|V|^2}{|V_0|^2}$$

so that:

$$\prod_{i=1}^{n-1} \lambda_i = \det(g_a) \frac{|V|^2}{|V_0|^2} = \frac{|V|^2}{|V_0|^2} e^{\lambda u} |V|^2_0$$

and:

$$\sum_{i=1}^{n-1} \lambda_i = \sum_{i=1}^{n} \lambda_i - \lambda_n = n + \Delta_0 u - \frac{|V|^2}{|V_0|^2} = m + \Delta_0^H u$$

The following is the main result of this section:

**Theorem 5.6.** The following differential inequality holds for solutions to equation (7):

$$\hat{\Delta}_{e,a} \left( e^{-au}(\Delta_0^H u + m) \right) - 2A\beta JV \left( e^{-au}(\Delta_0^H u + m) \right) \geq c_1 \left( \frac{|V|^2}{|V_0|^2} e^{-F} \right) ^{\frac{1}{m-1}} e^{-au} (m + \Delta_0^H u)^{\frac{m}{m-1}} - \left( c_2 + \frac{c_2'}{|V|^2} \right) - (c_3 + \beta) e^{-au} (m + \Delta_0^H u).$$

where $\beta = \frac{c}{|V|^2(V_0^2+\epsilon)}$, $A = aJV(u)$ and where $c_1 := (a - C) \inf_M e^{-\frac{F}{m-1}} > 0$, $c_2 := \sup_M e^{-au} |\Delta_0^H F|$ and $c_2' := -||\nabla Z||_{L^2}^2$ are numbers that depend only on $a$ (which is any real number chosen so that $a - C > 0$), $||u||_{L^\infty(M)}$ and $\sup_M F$, the infimum of the horizontal bisectional curvature $C$:

$$C := \inf \hat{g}_{i\bar{j}} g_{0\bar{i}} R_{\bar{s}\bar{t}i\bar{k}}(g_0) \frac{\xi_i \xi_{\bar{k}} \eta_j \eta_{\bar{j}}}{|\xi|^2 |\eta|^2} = \inf_{\eta, \xi \in \mathcal{Q}_0} \frac{\text{Rm}(\eta, J\eta, \xi, J\xi)}{|\eta|^2 |\xi|^2}$$

and $c_3 := a \left( n - \frac{|V|^2}{|V_0|^2 + \epsilon} \right) \lambda$ is a number such that $ma := (n - 1) a \leq c_3 \leq n a$. 

\[ \square \]
Proof. We will resume the convention that, having fixed an arbitrary \( \epsilon > 0 \), we suppress the index \( \epsilon \) from all the quantities (e.g., \( \hat{g}^{ij}_u = \hat{g}^{ij}_u \), \( \hat{\Delta}_u \epsilon = \hat{\Delta}_u \), \( \nabla \hat{g}_{\epsilon u} = \nabla \hat{g}_u \) etc.).

It follows from inequality (36) of Proposition 5.3 and from Lemma 5.4 that:

\[
\begin{align*}
\hat{\Delta}_{u,\epsilon} \left( e^{-au}(\Delta^E_{0} u + n) \right) & \geq \\
& = e^{-au} \left( g^{ij}_0 g_{u,ij} g^{kl}_u \partial_i \partial_j g_{u,kl} + g^{ij}_0 g^{kl}_0 R_{ijkl}(g_0) g_{u,ijkl} + \Delta^E_{0} F - \frac{|\nabla Z|^2_{h_0}}{|V|^2_{0}} \right) \\
& \quad - ae^{-au}(\Delta^E_{0} u + m) \hat{\Delta}_u \epsilon u \\
& \quad - \frac{e^{-au}}{\Delta^E_{0} u + m} |\nabla g_{\epsilon, \epsilon}(\Delta^E_{0} u)|^2_{h_u} + 2 \beta JV(e^{-au})JV(\Delta^E_{0} u + m) \\
\end{align*}
\]

Next observe that:

\[
\frac{e^{-au}}{\Delta^E_{0} u + m} |\nabla g_{\epsilon, \epsilon}(\Delta^E_{0} u)|^2_{h_u} = \frac{e^{-au}}{\Delta^E_{0} u + m} h^{ij}_0 \hat{g}^{kl}_u u_{ki} g^{pq}_0 u_{pq,ij} 
\]

We first claim that:

\[
\hat{g}^{ij}_0 g^{kl}_u g^{ij}_u g_{u,kl} \partial_i \partial_j g_{u,ij} \partial_i \partial_j g_{u,ij} \geq \frac{1}{\Delta^E_{0} u + m} \hat{g}^{ij}_0 g^{kl}_0 u_{ki} g^{pq}_0 u_{pq,ij}. 
\]

In fact, we first notice that in eq. (45) we could have computed the operator \( \hat{\Delta}_u \epsilon \) with respect to a frame adapted to holomorphic coordinates \( w_1, \ldots, w_n \) and likewise the operator \( \Delta^E_{0} \) with respect to coordinates \( z_1, \ldots, z_n \). This results in modifying the expression of eq. (46) (yet to be proved) into the following form (this expression, naturally, can be proved directly by observing that changing the \( w \)’s into the \( z \)’s on either side of the proposed inequality, gives back the original one):

\[
\begin{align*}
\hat{g}_0(dz_i, dz_j) \hat{g}_u(dz_k, dz_l) \hat{g}_{u,\epsilon}(dw_i, dw_l) \\
\geq \frac{1}{\Delta^E_{0} u + m} \hat{g}_{u,\epsilon}(dw_i, dw_l) \hat{g}_0(dz_k, dz_l) \hat{g}_0(dz_p, dz_q) \\
\end{align*}
\]

From now on, \( \epsilon > 0 \) will be fixed. Notice that since \( \epsilon > 0 \), \( \hat{g}^{ij}_u := g^{ij}_u - \frac{Z_i Z_j}{|V|^2} \) is invertible (as it can be seen by performing the computation: \( \hat{g}^{ij}_u Z_i Z_j = |V|^2_{0} - \frac{|V|^2_{0}}{|V|^2_{0} + \epsilon} > 0 \), where \( Z, dz_i \) is the holomorphic 1-from such that \( g^{ij}_0 Z_i Z_j = |V|^2_{0} \). For the sake of brevity and ease of notation, we will denote the indexes related to the coordinates \( z_1, \ldots, z_n \) with latin characters and the ones related to \( w_1, \ldots, w_n \)
with greek letters, so that we may write the equation above as:

\[
\hat{g}^{ij} \alpha \beta \frac{1}{\Delta_0 u + m} \hat{g}^{k\ell} \alpha \beta \partial_i g_{kl\bar{\alpha}} \bar{\partial_j} g_{u\bar{\alpha}} + R(\delta) \geq \Delta H_0 u + m \hat{g}_{\alpha \beta} u, \epsilon \hat{g}^{pq} u_{pq\beta}.
\]

(48)

In order to prove the inequality we introduce an auxiliary parameter \(\delta > 0\) and prove the following inequality instead:

\[
\hat{g}^{ij} \alpha \beta \frac{1}{\Delta_0 u + m} \hat{g}^{k\ell} \alpha \beta \partial_i g_{kl\bar{\alpha}} \bar{\partial_j} g_{u\bar{\alpha}} + R(\delta) \geq \Delta H_0 u + m \hat{g}_{\alpha \beta} u, \epsilon \hat{g}^{pq} u_{pq\beta}.
\]

(48)

where we have denoted (consistently with the notation used so far) \(g^{ij}_{\delta} := g^{ij} - \frac{Z_{\delta} Z_{\bar{\delta}}}{|V|_{0}^{2} + \delta} \) and \(g^{ij}_{\alpha} := g^{ij} - \frac{Z_{\delta} Z_{\bar{\delta}}}{|V|_{0}^{2} + \delta} \). Here:

\[
R(\delta) := \frac{1}{\Delta_0 u + n} \frac{\delta}{|V|_{0}^{2} + \delta} \hat{g}^{ij}_{\alpha, \epsilon} \left( \frac{Z \bar{Z}(u)}{|V|_{0}^{2}} \right) \left( \frac{Z \bar{Z}(u)}{|V|_{0}^{2}} \right)_{j}
\]

and is such that \(|R(\delta)| \to 0\) ad \(\delta \to 0\) (having fixed the solutions of the \(\epsilon\)-perturbed equation, with \(\epsilon > 0\)). Before we prove (48), we remark that taking the limit for \(\delta \to 0\) (which exists for fixed \(\epsilon > 0\)) of this inequality, we get back (61).

We now observe that (having fixed \(\epsilon > 0\) and \(\delta > 0\)) we may choose holomorphic coordinates, at any given point \(x \in M, z_1, \ldots, z_n\) such that \(g^{ij}_{\delta}\) and \(g^{ij}_{\alpha}\) are simultaneously diagonalized, say \(g^{ij}_{\delta}(x) = \mu^i \delta_{ij}\) and \(g^{ij}_{\alpha}(x) = \lambda^i \delta_{ij}\). We may also choose the coordinates \(w_1, \ldots, w_n\) so that \(g^{ij}_{\alpha, \beta}\) is diagonal, say \(g^{ij}_{\alpha, \epsilon}(x) = \Lambda_{\alpha} \delta_{ij}\).

Let us denote by \(\lambda^i\) the eigenvalue of \(g^{ij}_{\alpha, \beta}\) that goes to zero as \(\delta \to 0\) and \(\mu^i\) the one of \(g^{ij}_{\delta, \alpha}(x)\) that goes to zero.

In these coordinates:

\[
R(\delta) := \frac{1}{\Delta_0 u + n} \mu^i \sum_{\alpha} \Lambda_{\alpha} \left| u_{m\alpha} \right|^2
\]

Then a simple application of the Cauchy-Schwarz inequality yields:
\[
\frac{1}{\Delta_0^H u + m} \hat{g}_u^{\alpha \beta} \hat{g}^{\dot{k} \dot{I}}_0 u_{k \alpha} \hat{g}^{\dot{p} \dot{q}}_0 u_{p \dot{q} \bar{\beta}} = \frac{1}{\Delta_0^H u + m} \sum_{\alpha, k, \ell, \alpha = 1}^n \Lambda_{\alpha} \mu^k \mu^\ell u_{k \alpha} u_{\ell \bar{\alpha}}
\]

\[
\leq \frac{1}{\Delta_0^H u + m} \sum_{\alpha, k, \ell, \alpha = 1}^n \Lambda_{\alpha} \mu^k |u_{k \alpha}|^2 = \frac{1}{\Delta_0^H u + m} \sum_{\alpha = 1}^n \sum_{k = 1}^{n-1} \Lambda_{\alpha} \mu^k \left( \frac{u_{k \alpha} \lambda^k}{\lambda^2} \right) \left( u_{k \alpha} \lambda^{k \bar{2}} \right)
\]

\[
+ \frac{1}{\Delta_0^H u + m} \mu^n \sum_{\alpha} \Lambda_{\alpha} |u_{n \alpha}|^2
\]

\[
\leq \frac{1}{\Delta_0^H u + m} \left( \sum_{k, i, \alpha = 1}^n \mu^i \Lambda_{\alpha} \lambda^k u_{k \alpha} u_{\bar{k} \bar{\alpha}} \right) \left( \sum_{l = 1}^{n-1} \lambda^{l-1} \right) + \frac{1}{\Delta_0^H u + m} \mu^n \sum_{\alpha} \Lambda_{\alpha} |u_{n \bar{\alpha}}|^2
\]

\[
= \frac{\Delta_0^H u + m}{\Delta_0^H u + m} \left( g_0^{ij} \hat{g}^{\dot{i} \dot{j} \alpha \beta} \hat{g}^{\dot{k} \dot{l} \alpha \beta} R_{\alpha \beta \dot{I} \dot{J}} g_0^{\dot{I} \dot{J}} \lambda^{k \bar{2}} \right) g_{a \bar{b}, \delta} g_{a \bar{b} \delta} + R(\delta)
\]

In the last equality, we have used that \( \sum_{l = 1}^{n-1} \lambda^{l-1} = \text{tr}_{g_0} \hat{g}_{a \delta} \) where \( \hat{g}_{a \delta} \) is the inverse of \( g_{a \delta} \). and such inverse can be calculated in a coordinate system where \( g_{a \bar{b}} \) is diagonal, \( g_{0a} = 1 \) and \( Z^k(p) = 0 \) for \( k < n \). In such coordinates \( \text{tr}_{g_0} g_{a \delta} = \sum g_0^{i \bar{j}} \hat{g}_{i \bar{j}} = \Delta_0^H u + m \).

Alternatively, one could use the fact that \( \hat{g}_{a \epsilon} > 0 \) (at least at any point \( p \)) such that \( Z(p) \neq 0 \), but this assumption is certainly not restrictive) for any \( \epsilon > 0 \) and then, after observing that if \( b := \frac{e^{V_0^2}}{2|V_0|+e} \), then \( \hat{g}_{a \epsilon} + b \hat{g}_{a \epsilon} > 0 \), one may use the elementary Linear Algebra fact that if \( Q_1, Q_2 \) are Hermitian matrices and \( Q_1 > 0 \), then they can be simultaneously diagonalized (as in fact, \( Q_2 := Q_1 + a Q_2 > 0 \) for some \( a \in \mathbb{R} \) and clearly if both \( Q_1 \) and \( Q_2 \) are diagonalized simultaneously, then so are \( Q_1 \) and \( Q_2 \)).

In light of equation (46), eq. (45) becomes:

\[
\tilde{\Delta}_{u, \epsilon} (e^{-au}(\Delta_0^H u + m)) - 2 A \beta J V (e^{-au}(\Delta_0^H u + m)) \geq e^{-au} \Delta_0^H F + e^{-au} \hat{g}^{\dot{i} \dot{j}}_0 \hat{g}^{\dot{I} \dot{J}}_0 R_{\alpha \beta \dot{I} \dot{J}} (g_0) g_{a \bar{b}, \delta} \hat{g}_{a \bar{b} \delta} - a e^{-au}(\Delta_0^H u + m) \tilde{\Delta}_{u, \epsilon} u.
\]

\[
+ 2 \beta J V ( e^{-au} ) J V ( \Delta_0^H u + m ) - \frac{\nabla Z |^2 e_0}{|V_0|^2} - 2 A \beta J V ( e^{-au}(\Delta_0^H u + m))
\]

Next observe that:

\[
2 \beta J V ( e^{-au} ) J V (\Delta_0^H u + m) - 2 A \beta J V ( e^{-au}(\Delta_0^H u + m))
\]

\[
= -2 a \beta J V (u) J V (\Delta_0^H u + m) e^{-au}
\]

\[
+ 2 A a \beta J V (u)(\Delta_0^H u + m) e^{-au} - 2 A \beta J V ( (\Delta_0^H u + m) ) e^{-au}
\]

\[
= -2 a^2 \beta J V (u)^2 (\Delta_0^H u + m) e^{-au}
\]

\[4\text{Simultaneous diagonalization here is of course taken to mean that there is a matrix } M \text{ such that } M^* Q_1 M \text{ and } M^* Q_2 M \text{ are both diagonal.}\]
having chosen $A = -aJV(u)$. We now set:

$$C := \inf g_{ij}g_{0j}g_{0i}R_{ijkl}(g_0) \frac{\xi_i \xi_j \eta_{ik} \eta_{lj}}{||\xi_i ||^2 ||\eta_j ||^2}$$

for any vector field $\eta \neq 0$ and 1-form $\xi \neq 0$.

Therefore, in coordinates where $\hat{g}^{kl}(p) = \hat{\lambda}_k \delta_{kl}$ and $\hat{g}_{uij}(p) = \mu_i \delta_{ij}$, using that $\lambda_k, \mu_i \in \mathbb{R}$ and that $\mu_n = 0$ (by Lemma 3.1):

$$\hat{g}^{kl} g_{0k} g_{0l} R_{ijkl}(g_0) \hat{g}_{uij} = \sum_{k=1}^{n} \sum_{i=1}^{n} \hat{g}^{kl} g_{0k} g_{0l} R_{ijkl}(g_0) \hat{\lambda}_k \mu_i$$

$$\geq -C(\hat{g}_{0i} \mu_i)(g_{0kk} \hat{\lambda}_k) = -C(\Delta_0^H u + m)(\hat{g}_{ui} g_{0k} \hat{\lambda}_k)$$

Since:

$$\hat{g}_{ui} g_{0k} \hat{\lambda}_k = \hat{g}_{ui} g_{0k} + \beta Z^k \hat{Z}^l g_{0kl} = \hat{g}_{ui} g_{0k} + \beta |V|^2_0$$

the previous equation becomes:

$$\hat{g}^{kl} g_{0k} g_{0l} R_{ijkl}(g_0) \hat{g}_{uij} \geq -C(\Delta_0^H u + m)\hat{g}^{kl} g_{0k} - C \beta |V|^2_0 (\Delta_0^H u + m)$$

On the other hand, since $\hat{\Delta}_{u,\epsilon} u = \hat{g}_{ui,\epsilon} u_{ij}$, it follows that:

$$\hat{\Delta}_{u,\epsilon} u = \hat{g}_{ui,\epsilon} (g_{ui,j} - g_{0ij}) = \left( g_{ui} - \frac{Z^l \hat{Z}^j}{|V|^2_0 + \epsilon} \right) g_{uj} - \hat{g}^{ij} g_{0ij}$$

$$= \text{tr}_u g_u - \frac{Z^l \hat{Z}^j}{|V|^2_0 + \epsilon} g_{0ij} - \hat{g}^{ij} g_{0ij}$$

$$= n - \frac{|V|^2_0}{|V|^2_0 + \epsilon} - \hat{g}^{ij} g_{0ij}.$$}

It follows that (in coordinates as above) (and with $g_{0\alpha} = 1$ at the point):

$$\hat{\Delta}_{u,\epsilon} u = n - \frac{|V|^2_0}{|V|^2_0 + \epsilon} - \hat{g}^{ij} g_{0ij} = n - \frac{|V|^2_0}{|V|^2_0 + \epsilon} - \sum_{i=1}^{n-1} \lambda_i g_{0\alpha} = n - \frac{|V|^2_0}{|V|^2_0 + \epsilon} - \sum_{i=1}^{m} \lambda_{ij}$$

which yields:

$$- a e^{-au} \left( m + \Delta_0^H u \right) \hat{\Delta}_{\epsilon, u} \hat{\Delta}^\beta u$$

$$= a \sum_{i=1}^{n} \lambda_i - a e^{-au} \left( n - \frac{|V|^2_0}{|V|^2_0 + \epsilon} \right) (m + \Delta_0^H u).$$

Observe that:

$$m := n - 1 \leq n - \frac{|V|^2_0}{|V|^2_0 + \epsilon} \leq n.$$
Setting $C_F := \sup_M |\Delta^H_0 F|$ and putting everything together back into equation (50):

$$
\hat{\Delta}_{u,\epsilon} \left( e^{-au} (\Delta^H_0 u + m) \right) \geq e^{-au} (a - C) \left( n + \Delta^H_0 u \right) g^{kl} g_{0k\bar{l}} - e^{-au} C_F
- a e^{-au} \left( n - \frac{|V|^2_u}{|V|^2_u + \epsilon} + C \beta |V|^2_0 + 2a\beta \right) (m + \Delta^H_0 u) - \frac{|\nabla Z|^2_{h_0}}{|V|^2_0}.
$$

(51)

On the other hand, since at the given point:

$$
\prod_{i=1}^{n-1} \lambda_i^{-1} = \det(g_u) \frac{|V|^2_u}{|V|^2_u} = \frac{|V|^2_u + \epsilon}{|V|^2_u} e^F |V|^2_0
$$

and since by the arithmetic/geometric mean inequality:

$$
\sum_{i=1}^{n-1} \lambda_i \geq \left( \frac{\sum_{i=1}^{n-1} \lambda_i^{-1}}{\prod_{i=1}^{n-1} \lambda_i^{-1}} \right)^{\frac{1}{m-1}} \left( \frac{|V|^2_u}{(|V|^2_u + \epsilon)|V|^2_0} \right)^{\frac{1}{m-1}} e^{-\frac{F}{m-1}} \left( m + \Delta^H_0 u \right)^{\frac{1}{m-1}}
$$

from equation (51), if we choose $a$ such that $a > C$, we obtain:

$$
\hat{\Delta}_{u,\epsilon} \left( e^{-au} (\Delta^H_0 u + m) \right) \geq e^{-au} (a - C) \left( \frac{|V|^2_u e^{-F}}{|V|^2_0} \right) \left( m + \Delta^H_0 u \right)^{\frac{1}{m-1}} (m + \Delta^H_0 u)
- e^{-au} C_F - a e^{-au} \left( n - \frac{|V|^2_u}{|V|^2_u + \epsilon} + C \beta |V|^2_0 \right) (m + \Delta^H_0 u)
- \frac{|\nabla Z|^2_{h_0}}{|V|^2_0}.
$$

(52)

and the Theorem is proved. \qed

We now need to deal with the term $\frac{\epsilon^2}{|V|^2_0}$. As a preliminary step we have:

**Lemma 5.7.** For any real number $\alpha$, $0 < \alpha \leq 1$:

$$
\hat{\Delta}_{u,\epsilon} (|V|^2_0) - \frac{1}{|V|^2_0} \frac{|\nabla_{u,\epsilon} V|^2_0}{|V|^2_0} \geq -C_0 |V|^2_0 (\tilde{g}_{0i} g_{0j})
$$

(53)

where $C_0 := \sup_{|v| = 1} \text{Rm}(g_0)(v, Jv, \frac{V}{|V|^0}, \frac{JV}{|V|^0})$ (which we may assume, without loss of generality, to be the same as the constant $C$ in eq. (52)).

**Proof.** The complex Hessian of $|Z|^2_0$ equals:

$$
\frac{\partial^2}{\partial z_i \partial \bar{z}_j} |Z|^2_0 = -\text{Rm}(g_0)_{ijkl} Z^k \bar{Z}^l + g_{0k\bar{l}} \frac{\partial Z^k}{\partial z_i} \frac{\partial \bar{Z}^l}{\partial \bar{z}_j}
$$

We then choose normal holomorphic coordinates in which $g_{0i}$ is diagonal and $Z^k(p) = 0$ for $k < n$, so that $\partial_i |V|^2_0 = g_{0i} \frac{\partial}{\partial z_i} |V|^2_0 = Z^i Z^n$ to obtain:
\[ \hat{g}^{ij}_{u,e} \frac{\partial Z^k}{\partial z_i} \frac{\partial \tilde{Z}^l}{\partial z_j} = \frac{\left| \nabla_u T_{ij} V_0^2 \right|^2}{|V_0^2|} \]
\[ = \hat{g}^{ij}_{u,e} \left( \sum_{k=1}^{n} |Z_i^k|^2 - \frac{g_0^{ij} Z_i^g Z_j^i \tilde{Z}_j}{|Z_n^i|^2} \right) = \hat{g}^{ij}_{u,e} \left( \sum_{k=1}^{n} |Z_i^k|^2 - Z_i^g \tilde{Z}_j^i \right) \geq 0 \] (54)

This readily yields:
\[ \hat{\Delta}_{u,e} \left( |V_0^2| \right) - \frac{\left| \nabla_u V_0^2 \right|^2}{|V_0^2|} u = \hat{g}^{ij}_{u,e} g_{ijkl} \left( -Rm(g_0)_{ijkl} Z_k \tilde{Z}_l + g_{ijkl} \frac{\partial Z_k}{\partial z_i} \frac{\partial \tilde{Z}_l}{\partial z_j} \right) \]
\[ - \frac{\left| \nabla_u V_0^2 \right|^2}{|V_0^2|} u \geq \sum_{i=1}^{n-1} \lambda_i \left( -Rm(g_0)_{ijkl} |Z_n|^2 \right) \geq -C_0 |V_0^2| \sum_{i=1}^{n-1} \lambda_i \] (55)

where \( C_0 := \sup_{x,k} \hat{g}^{ij}_{00} R(g_0)_{ijkl} Z^n \tilde{Z}^l \) (which we may assume, without loss of generality, to be the same as the constant \( C \) in eq. (52)).

We are now ready to prove:

**Theorem 5.8.** The following differential inequality holds for solutions to equation (7). Let \( y_a := |V_0^{2a} \left( e^{-au} (\Delta^H u + m) \right) \), then:
\[ \hat{\Delta}_{e,u} (y_a) - 2 A \beta J V (y_a) \geq c_1 \left( \frac{|V_0^2 e^{-F}|}{(|V_0^2| + \epsilon)} \right)^{-\frac{1}{m-1}} \left( y_a \right)^{\frac{m}{m-1}} \left( \frac{1}{|V_0^{2a}|} - \frac{a+1}{m-1} - (c_2 |V_0^{2a} + c_2' |V_0^{2a-2} \right) \] (56)

where \( \beta = \frac{e^{-F}}{|V_0^{2a}| (|V_0^{2a} + \epsilon) \}, A = a J V (u) \) and where \( c_1 := (a - C) \inf_M e^{-F} \), \( c_2 := a \sup_M e^{-au} |\Delta^H F| \) and \( c'_2 := -\| \nabla Z \|_{L_0^2} \) are numbers that depend only on \( a \) (which is any real number chosen so that \( a - C > 0 \), \( |u| L_\infty (M) \) and \( F \), the infimum of the horizontal bisectional curvature \( C \):
\[ C := \inf \hat{g}^{ij}_{00} R_{ijkl}(g_0) \frac{\xi_i \xi_j \eta_k \eta_l}{|\xi|^2 |\eta|^2} = \inf_{\eta \in Q_0} \frac{\text{Rm}(\eta, J(\eta, \xi), J(\xi))}{|\eta|^2 |\xi|^2} \]
and \( c_3 := a \left( n - \frac{|V_0^2|}{|V_0^{2a}|} \right) \) is a number such that: \( m a := (n - 1) a \leq c_3 \leq n a \).

**Proof.** From now on let us set:
\[ y_H := e^{-au} (n - 1 + \Delta^H u) \]

We can therefore rewrite eq. (51) as:
\[ \hat{\Delta}_{u,e} y_H - 2 A \beta J V (y_H) \geq (a - C) y_H \sum_{i=1}^{n-1} \lambda_i - c_2 y_H - \left( c_3' + \frac{c_3''}{|V_0^2|} \right) \] (57)
Next, one readily computes:
\[
\left(\hat{\Delta}_{u,\epsilon} - 2A\beta J\nu \right) \left( |V|^{2\alpha} e^{-au}(\Delta^H_0 u + m) \right) = |V|^{2\alpha} e^{-au} \hat{\Delta}_{u,\epsilon}(\Delta^H_0 u + m) \\
+ \nabla_{g_{u,\epsilon}}(\Delta^H_0 u + m) \cdot \left( |V|^{2\alpha} \nabla_{g_{u,\epsilon}}(e^{-au}) + \nabla_{g_{u,\epsilon}}(|V|^{2\alpha}) e^{-au} \right) \\
+ 2A\beta J\nu \left( e^{-au}(m + \Delta^H_0 u) \right) J\nu(|V|^{2\alpha}) \\
+ (\Delta^H_0 u + m) \left[ |V|^{2\alpha} e^{-au} \left( -a \hat{\Delta}_{u,\epsilon} u + a^2 |\nabla_{g_{u,\epsilon}} u|_{g_{u,\epsilon}}^2 \right) + e^{-au} \left( \hat{\Delta}_{u,\epsilon} - 2A\beta J\nu \right) |V|^{2\alpha} \right] 
\]

Direct computations yield:
\[
\hat{\nabla}_{u,\epsilon}|V|^{2\alpha}_0 = \alpha |V|^{2(\alpha-1)}_0 \hat{\nabla}_{u,\epsilon}|V|^{2}_0 \\
\]
and:
\[
\hat{\Delta}_{u,\epsilon}(|V|^{2\alpha}_0) = \alpha |V|^{2(\alpha-1)}_0 \hat{\Delta}_{u,\epsilon}|V|^{2}_0 + \alpha(\alpha - 1)|V|^{2(\alpha-2)}_0 \left| \hat{\nabla}_{u,\epsilon}|V|^{2}_0 \right|^2 
\]

Next observe that by Cauchy-Schwartz one has:
\[
- a|V|^{2\alpha}_0 \nabla_{g_{u,\epsilon}}(\Delta^H_0 u) \cdot \nabla_{g_{u,\epsilon}}(u) + \alpha |V|^{2(\alpha-1)}_0 \nabla_{g_{u,\epsilon}}(\Delta^H_0 u) \cdot \nabla_{g_{u,\epsilon}}(|V|^{2}_0) \\
\geq - a|V|^{2\alpha}_0 \nabla_{g_{u,\epsilon}}(\Delta^H_0 u) \cdot \nabla_{g_{u,\epsilon}}(u) - \alpha |V|^{2(\alpha-1)}_0 \left| \nabla_{g_{u,\epsilon}}(\Delta^H_0 u) \cdot \nabla_{g_{u,\epsilon}}(|V|^{2}_0) \right| \\
\geq - |V|^{2\alpha}_0 \left( \frac{|\nabla_{g_{u,\epsilon}}(\Delta^H_0 u)|^2_{g_{u,\epsilon}}}{\Delta^H_0 u + m} + a^2 |\nabla_{g_{u,\epsilon}} u|_{g_{u,\epsilon}}^2 (\Delta^H_0 u + m) \right) \\
- |V|^{2(\alpha-1)}_0 \left( \frac{|\nabla_{g_{u,\epsilon}}(\Delta^H_0 u)|^2_{g_{u,\epsilon}}}{\Delta^H_0 u + m} + \alpha^2 \frac{|\nabla_{g_{u,\epsilon}}|V|^{2}_0|^2_{g_{u,\epsilon}}}{|V|^{2}_0} (\Delta^H_0 u + m) \right) 
\]

Therefore, going back to (58) and using (59):
\[
\hat{\Delta}_{u,\epsilon} \left( |V|^{2\alpha}_0 e^{-au}(\Delta^H_0 u + m) \right) \geq |V|^{2\alpha}_0 e^{-au} \hat{\Delta}_{u,\epsilon}(\Delta^H_0 u + m) \\
- 2 |V|^{2\alpha}_0 \frac{|\nabla_{g_{u,\epsilon}}(\Delta^H_0 u)|^2_{g_{u,\epsilon}}}{\Delta^H_0 u + m} - a|V|^{2\beta}_0 e^{-au} \hat{\Delta}_{u,\epsilon} u \\
+ \alpha e^{-au}(\Delta^H_0 u + m) |V|^{2(\alpha-1)}_0 \left( \hat{\Delta}_{u,\epsilon} |V|^{2\alpha}_0 - \frac{|\nabla_{u,\epsilon}|V|^{2\alpha}_0|^2}{|V|^{2\alpha}_0} \right) 
\]

Reviewing equation (49) with a keener eye (reading just the first and penultimate lines), we observe that it reads:
\[
2 \frac{1}{\Delta^H_0 u + m} \nabla_{g_{u,\epsilon}} g_{0,k\alpha} g_{0,k\alpha} e^{-ap} u_{p\beta} \\
\leq 2 \frac{1}{\Delta^H_0 u + m} \left( \sum_{k,\alpha=1}^n \mu^k \Lambda^k \Lambda^k u_{k\alpha} u_{k\alpha} \right) \left( \sum_{i=1}^{n-1} \lambda^i \right) + \frac{2}{\Delta^H_0 u + m} \mu^n \sum_{\alpha=1}^n \Lambda^\alpha \left| u_{\alpha\alpha} \right|^2 
\]
and therefore, after observing that:

\[
2 \sum_{k,\alpha=1}^{n} \mu^k \Lambda^\alpha \chi^k u_{k\alpha k} u_{\alpha k} \leq \sum_{k,\alpha=1}^{n} \mu^k \Lambda^\alpha \chi^k u_{k\alpha i} u_{\alpha k i}
\]
the analogous of equation (46) reads:

\[
\begin{align*}
\sum_{i,j,k,l} g_{0} g_{u} \bar{g}_{u,\epsilon} \bar{g}_{u,\epsilon} \bar{g}_{u,\epsilon} \bar{g}_{u,\epsilon} & \sum_{k,l} \mu^k \Lambda^\alpha \chi^k u_{k\alpha k} u_{\alpha k} \\
\geq & \frac{2}{\Delta^H u + m} \sum_{i,j,k,l} g_{0} g_{u} \bar{g}_{u,\epsilon} \bar{g}_{u,\epsilon} \bar{g}_{u,\epsilon} \bar{g}_{u,\epsilon} \mu^i \Lambda^\alpha \chi^i u_{k\alpha i} u_{\alpha k i}
\end{align*}
\]

(61)

Whence, appealing to equations (45), (61) and (60) (and using the same notation as in equation (57)):

\[
\begin{align*}
\left( \Delta_{u,\epsilon} - 2A \beta J \right) \left( e^{-\alpha u (m + \Delta^H u)} \right) J V (|V|^{2\alpha}_0 y_H) & \geq \\
\left( \alpha - C \right) y_H \sum_{i=1}^{n-1} \lambda_i - c_2 y_H - \left( c^{\prime 3}_3 + \frac{c^{\prime 3}_3}{|V|^{2\alpha}_0} \right) |V|^{2\alpha}_0 & \geq \\
\alpha y_H |V|^{2(\alpha - 1)}_0 & \left( \Delta_{u,\epsilon} |V|^{2\alpha}_0 - \frac{|\widehat{V}_{u,\epsilon}|V|^{2\alpha}_0}{|V|^{2\alpha}_0} \right)
\end{align*}
\]

(62)

Putting together equations (62) and (53) in Lemma 5.7 yields:

\[
\begin{align*}
\left( \Delta_{u,\epsilon} - 2A \beta J \right) |V|^{2\alpha}_0 y_H & \geq \\
\left( \alpha - (1 + \alpha) C \right) y_H \sum_{i=1}^{n-1} \lambda_i - c_2 y_H - \left( c^{\prime 3}_3 + \frac{c^{\prime 3}_3}{|V|^{2\alpha}_0} \right) |V|^{2\beta}_0 & \geq \\
\alpha y_H & |V|^{2(\alpha - 1)}_0
\end{align*}
\]

(63)

Therefore choosing \( a = -(1 + \alpha) C > 0 \) and using the arithmetic/geometric mean inequality as in the derivation of equation (51), yields:

\[
\begin{align*}
\left( \Delta_{u,\epsilon} - 2A \beta J \right) (y_H |V|^{2\alpha}_0) & \geq \left( \alpha - (1 + \alpha) C \right) y_H \frac{1}{m-1} \left( \frac{|V|^{2\alpha}_u e^{-F}}{|V|^{2\alpha}_0 (|V|^{2\alpha}_u + \epsilon)} \right)^{\frac{1}{m-1}} |V|^{2\alpha}_0 - c_2 |V|^{2\alpha}_0 y_H \\
& - (c^{\prime 3}_3 |V|^{2\alpha}_0 + c^{\prime 3}_3 |V|^{2\alpha - 2}_0)
\end{align*}
\]

(64)

Which we can then rewrite as:

\[
\begin{align*}
\left( \Delta_{u,\epsilon} - 2A \beta J \right) (y_H |V|^{2\alpha}_0) & \geq c_1 \frac{|V|^{2\alpha}_u}{|V|^{2\alpha}_0 + \epsilon} y_H \frac{1}{m-1} \left( \frac{1}{|V|^{2\alpha}_0} \right)^{\frac{1}{m-1}} |V|^{2\alpha}_0 - c_2 |V|^{2\alpha}_0 y_H \\
& - (c^{\prime 3}_3 |V|^{2\alpha}_0 + c^{\prime 3}_3 |V|^{2\alpha - 2}_0)
\end{align*}
\]

(65)
with $c_1 > 0$. After a straightforward algebraic manipulation, we can rewrite this equation as:

$$
\left( \hat{\Delta}_{u, \epsilon} - 2A \beta J V \right)(y_H |V|^{2\alpha}_0) \geq c_1 \left( y_H |V|^{2\alpha}_0 \right)^{-\frac{\alpha+1}{m-1}} - c_2 y_H |V|^{2\alpha}_0
$$

and

$$
\left( \hat{\Delta}_{u, \epsilon} - 2A \beta J V \right)(y_H |V|^{2\alpha}_0) \geq c_1 \left( y_\alpha |V|^{2\alpha}_0 \right)^{-\frac{\alpha+1}{m-1}} - c_2 y_\alpha
$$

Setting $y_\alpha := |V|^{2\alpha}_0 y_H$, we may then succinctly write the former as:

$$
\left( \hat{\Delta}_{u, \epsilon} - 2A \beta J V \right)(y_H |V|^{2\alpha}_0) \geq c_1 (y_\alpha)^{-\frac{\alpha+1}{m-1}} - c_2 y_\alpha
$$

An application of the proof of Theorem 5.6 is the following theorem whose main importance stems from the fact that one can get better derivative estimates under stronger assumptions (which include the case of the classical Ricci flow), since the term $- \frac{\nabla V}{V^{\alpha}_0}$ is not present.

**Theorem 5.9.** Keeping the same notation as above. Suppose $g_\alpha$ is a solution to the $\epsilon$ perturbed $V$-soliton equation (i.e., eq. (8)). Assume further that there is a background metric $g_0$ such that:

$$
JV(h_0^{ij}) = B h_0^{ij}
$$

for some function $B$ with $\|B\|_\infty \leq b$ for some uniform $b \in \mathbb{R}_+$. Then:

$$
\hat{\Delta}_{\epsilon, \omega} \left( e^{-\alpha u}(\Delta_0^H u + m) \right) - 2aJV(u)\beta JV \left( e^{-\alpha u}(\Delta_0^H u + m) \right) \geq
$$

$$
c_1 e^{-\alpha u} (m + \Delta_0^H u)^{\frac{\alpha}{m-1}} - (c_2 + B^2 n \beta) - (c_3 + \beta B) e^{-\alpha u}(m + \Delta_0^H u).
$$

where $c_1 := (a - C) \inf_M e^{-\frac{\epsilon F}{2}} > 0$, $c_2 := a \sup_M e^{-\alpha u} |\Delta_0 F|$, are numbers that depend only on $a$ (which is any real number chosen so that $a - C > 0$), $\|u\|_{L^\infty(M)}$ and $\sup_M F$, the infimum of the horizontal bisectional curvature $C$:

$$
C := \inf \hat{g}_0^{ij} \hat{g}_0^{k\ell} R_{ijkl}(g_0) \xi_j \xi_k \eta^i \eta^j,\eta^{k\ell} \xi^i \xi^k \eta^j \eta^{k\ell}
$$

and $c_3 := a \left( n - \frac{|V|^{2\alpha}_0}{|V|^{\alpha}_0 + \epsilon} \right)$.

**Proof.** This follows from the differential inequality (52) since the assumption of $JV(h_0^{ij})$ yields:

$$
2 \beta JV(h_0^{ij})JV(u_{ij}) = 2 \beta B h_0^{ij} JV(u_{ij}) = 2 \beta B JV(\Delta_0^H u + m) - 2 \beta B JV(h_0^{ij})u_{ij}
$$

$$
= 2 \beta B JV(\Delta_0^H u + m) - 2 \beta B^2 \Delta_0^H u
$$

**Remark 5.10.** We remark that in the interesting case in which the $V$-soliton equation arises from the solution of the Ricci flow on a given manifold $X$, we take $M$ to be of the form: $M = X \times S^2 \times \mathbb{R}_+$ endowed with background metric
\[ g_0 = h_0 + w_0 d\tau_0^2 + \frac{1}{w_0} \theta^2 \] with \( \frac{\partial h}{\partial \tau_0} = 0 \). Then the preceding Theorem can be applied to reprove the well known regularity of the Kähler-Ricci flow (since \( JV(h_0) = 0 \) so we can take \( B = 0 \)).

5.2. **Two more important differential inequalities.** Here we prove a differential inequality involving \((n + \Delta_0 u)\).

We will start with:

**Theorem 5.11.** One has:

\[
\hat{\Delta}_{a, \epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq (a - C) \left( e^{-au}(n + \Delta_0 u) \right) \text{tr}_{g_0} \left( g_0 \right) \\
- a \left( n - \frac{|V|_0^2}{|V|_0^2 + \epsilon} + \frac{|V|_0^2}{|V|_0^2 + \epsilon} e^{-au}(n + \Delta_0 u) \right) \\
- \|e^{-au}\|_\infty \left( \|R(g_0)\|_\infty + \|\Delta_0 F\|_\infty \right) 
\]

where \( C := \inf_{(v, w)}: \|v\| = \|w\| = 1 \) \( Rm(v, Jv, w, Jw) \).

In particular:

\[
\hat{\Delta}_{a, \epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq (a - C) \left( e^{-au}(n + \Delta_0 u) \right) \text{tr}_{g_0} \left( g_0 \right) \\
- c_2 e^{-au}(n + \Delta_0 u) \\
- \|e^{-au}\|_\infty \left( \|R(g_0)\|_\infty + \|\Delta_0 F\|_\infty \right) 
\]

where:

\[ c_2 = c_2(a, n, |V|_0^2, |V|_0^2) := n + \frac{|V|_0^2}{|V|_0^2 + \epsilon} > 0 \]

**Proof.** We start from the standard:

\[ R(g_0)_{ijkl} = g^{pq}_{kl} \partial_i g_{aqk} \partial_j g_{up\bar{q}} + R(g_0)_{ijkl} - \partial_i \partial_j u_{kl} \]

and trace by \( \hat{g}^{k\bar{i}}_0 = g^{k\bar{i}}_0 - \frac{Z^k Z^l}{|V|_a^2 + \epsilon} \) and \( \hat{g}^{ij}_0 \) to obtain:

\[
- \hat{g}^{ij}_0 \frac{Z^k Z^l}{|V|_a^2 + \epsilon} R(g_0)_{ijkl} + \hat{g}^{ij}_0 R(g_0)_{ij} = \hat{g}^{ij}_0 \hat{g}^{k\bar{i}}_0 g^{pq}_{kl} \partial_i g_{aqk} \partial_j g_{pim}
\]

\[ + R(g_0)_{kl} \hat{g}^{k\bar{i}}_0 - \hat{g}^{ij}_0 \hat{g}^{k\bar{i}}_0 \partial_i \partial_j u_{kl} \]

On the other hand:

\[ \hat{\Delta}_{a, \epsilon} (\Delta_0 u) = \hat{g}^{k\bar{i}}_0 \hat{g}^{ij}_0 u_{ijkl} + \hat{g}^{k\bar{i}}_0 R(g_0)_{ijkl} u_{ij} - \hat{g}^{k\bar{i}}_0 R(g_0)_{kl} \]

thus:

\[
\hat{\Delta}_{a, \epsilon} (\Delta_0 u) = \hat{g}^{k\bar{i}}_0 \hat{g}^{ij}_0 g^{pq}_{kl} \partial_i g_{aqk} \partial_j g_{pim} + \hat{g}^{k\bar{i}}_0 R(g_0)_{ijkl} u_{ij} \\
- \hat{g}^{ij}_0 R(g_0)_{ij} + \hat{g}^{ij}_0 \frac{Z^k Z^l}{|V|_a^2 + \epsilon} R(g_0)_{ijkl} 
\]
Using the $\epsilon$-perturbed V-soliton equation (i.e., $R(g_u)_{ij} = \text{Ric}(g_0)_{ij} - \partial_i \partial_j (F + \log(|V|^2_u + \epsilon))$) and eq. (20) from Lemma 3.2:

\[-g_0^{ij} R(g_u)_{ij} + g_0^{ij} \frac{Z^k Z^l}{|V|^2_u + \epsilon} R(g_u)_{ijkl} = \]

\[-R(g_0) + \Delta_0 \left(F + \log(|V|^2_u + \epsilon)\right) + g_0^{ij} \frac{Z^k Z^l}{|V|^2_u + \epsilon} R(g_u)_{ijkl} \geq -R(g_0) + \Delta_0 F\]

Whence:

$$\hat{\Delta}_{u,\epsilon} (\Delta_0 u) \geq g_0^{ij} g_u^{pq} \partial_i g_{uq} \partial_j g_{upl} + \hat{g}_u^{kl} R(g_0)_{ij} g_{uij} - R(g_0) + \Delta_0 F$$

Next we use the simple facts that:

$$\hat{\Delta}_{u,\epsilon} \left(e^{-au} (\Delta_0 u + n)\right) \geq e^{-au} \hat{\Delta}_{u,\epsilon} (\Delta_0 u + n) - ae^{-au} (\Delta_0 u + n) \hat{\Delta}_{u,\epsilon} u$$

and that:

$$\frac{e^{-au}}{\Delta_0 u + n} |\nabla_{g_{u,\epsilon}} (\Delta_0 u) |_{g_u} \leq \hat{g}_u^{kl} g_u^{pq} \partial_i g_{uq} \partial_j g_{upl}$$

to conclude that:

$$\hat{\Delta}_{u,\epsilon} \left(e^{-au} (n + \Delta_0 u)\right) \geq e^{-au} \left(\hat{g}_u^{kl} R(g_0)_{ij} g_{uij} - R(g_0) + \Delta_0 F\right)$$

$$- ae^{-au} (\Delta_0 u + n) \hat{\Delta}_{u,\epsilon} u \tag{71}$$

Whence we get the conclusion by observing that if $C := \inf_{(v,w)}: ||v||=||w||=1 \ Rm(v, Jv, w, Jw)$ then:

$$\hat{g}_u^{kl} R(g_0)_{ij} g_{uij} \geq -C \left(\hat{g}_u^{kl} g_{okl}\right) \left( g_0^{ij} g_{uij} \right) =$$

$$- C \left(\hat{g}_u^{kl} g_{okl} - \frac{Z^k Z^l}{|V|^2_u + \epsilon} g_{uij}\right) (n + \Delta_0 u) =$$

$$- C \text{tr}_{g_u} (g_0)(n + \Delta_0 u) + C \left|\frac{|V|^2_u}{|V|^2_u + \epsilon}\right| \geq -C \text{tr}_{g_u} (g_0)(n + \Delta_0 u)$$

and that:

$$\hat{\Delta}_{u,\epsilon} u = \hat{g}_u^{ij} (g_{u,ij} - g_{0,ij}) = \left(\hat{g}_u^{ij} - \frac{Z^i Z^j}{|V|^2_u + \epsilon}\right) g_{uij} - \left(\hat{g}_u^{ij} - \frac{Z^i Z^j}{|V|^2_u + \epsilon}\right) g_{0,ij} =$$

$$\text{tr}_{g_u} g_{u,ij} - \frac{Z^i Z^j}{|V|^2_u + \epsilon} g_{uij} - \text{tr}_{g_u} (g_0) + \frac{Z^i Z^j}{|V|^2_u + \epsilon} g_{0,ij} =$$

$$n - \frac{|V|^2_u}{|V|^2_u + \epsilon} - \text{tr}_{g_u} (g_0) + \frac{|V|^2_0}{|V|^2_u + \epsilon}.$$
which is only slightly different from what was already observed in the proof of Theorem 5.6. Putting these together:

\[
\hat{\Delta}_{u,\epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq e^{-au} \left( -C \text{tr}_{g_u}(g_0)(n + \Delta_0 u) - R(g_0) + \Delta_0 F \right) \\
- ae^{-au}(\Delta_0 u + n) \left( n - \frac{|V|_u^2}{|V|_u^2 + \epsilon} - \text{tr}_{g_u}(g_0) + \frac{|V|_0^2}{|V|_u^2 + \epsilon} \right)
\]

and rearranging;

\[
\hat{\Delta}_{u,\epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq e^{-au}(a - C) \left( \text{tr}_{g_u}(g_0)(n + \Delta_0 u) \\
- ae^{-au}(\Delta_0 u + n) \left( n - \frac{|V|_u^2}{|V|_u^2 + \epsilon} + \frac{|V|_0^2}{|V|_u^2 + \epsilon} \right) \\
+ e^{-au}(-R(g_0) + \Delta_0 F) \right)
\]

Finally note that:

\[
\left( n - \frac{|V|_u^2}{|V|_u^2 + \epsilon} + \frac{|V|_0^2}{|V|_u^2 + \epsilon} \right) \leq n + \frac{|V|_u^2}{|V|_u^2 + \epsilon}
\]

We will also need:

Lemma 5.12. If \( g_u \) solves the \( \epsilon \)-perturbed \( V \)-soliton equation:

\[
\text{tr}_{g_u}(g_0) \geq \frac{e^{-\frac{F-\lambda u}{n}}}{(|V|_u^2 + \epsilon)^{\frac{1}{n}}} \left( e^{-au}(n + \Delta_0 u) \right)^{\frac{1}{2}}
\]

Proof. This is an easy consequence of the arithmetic/geometric mean inequality. Writing \( g_{0ij} = \delta_{ij} \) and \( g_{ui\bar{j}} = (1 + u_{ij})\delta_{ij} \) at a point \( p \), by the arithmetic/geometric mean inequality we have:

\[
\text{tr}_{g_u}(g_0) = \sum_{i=1}^{n} (1 + u_{ii}) \geq \left( \frac{\sum_{i=1}^{n} (1 + u_{ii})}{\prod_{i=1}^{n} (1 + u_{ii})} \right)^{\frac{1}{n}}
\]

and now, using that \( \sum_{i=1}^{n} (1 + u_{ii}) = n + \Delta_0 u \) and that, by the scalar \( \epsilon \)-perturbed \( V \)-soliton eq.:

\[
\prod_{i=1}^{n} (1 + u_{ii}) = (|V|_u^2 + \epsilon) e^{F-\lambda u}
\]

we are done. We can then prove:

Theorem 5.13.

\[
\hat{\Delta}_{u,\epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq (a - C) \left( e^{-au}(n + \Delta_0 u) \right) \left( \hat{g}_{u,\epsilon}^k \hat{g}_{0,k} \right) \\
- \left( n - \frac{|V|_u^2}{|V|_u^2 + \epsilon} \right) e^{-au}(n + \Delta_0 u) \\
- \| e^{-au} \|_\infty \left( \| R(g_0) \|_\infty + \| \Delta_0 F \|_\infty \right)
\]
where $C := \inf_{(v,w)}: \|v\| = \|w\| = 1 \ Rm(v, Jv, w, Jw)$.

Therefore:

\[
\hat{\Delta}_{u,e} \left( e^{-au}(n + \Delta_0 u) \right) \\
\geq (a - C) \ e^{-au}(n + \Delta_0 u) \left( \frac{|V|^2 |V|_u^2}{(|V|^2 + \epsilon) \det(g_0)} \right)^{\frac{1}{m-1}} e^{-\frac{F-\lambda u}{m-1}} (m + \Delta_0^H u)^{\frac{1}{m-1}} \\
- \left( n - \frac{|V|^2}{|V|^2 + \epsilon} \right) e^{-au}(n + \Delta_0 u) \\
- ||e^{-au}||_\infty (||R(g_0)||_\infty + ||\Delta_0 F||_\infty) \\
\]  

(74)

Proof. From equation (71) we have:

\[
\hat{\Delta}_{u,e} \left( e^{-au}(n + \Delta_0 u) \right) \geq e^{-au} \left( \tilde{g}^{kl} R(g_0) \tilde{g}_{uij} - R(g_0) + \Delta_0 F \right) \\
- ae^{-au}(\Delta_0 u + n) \hat{\Delta}_{u,e} u \\
\]  

(75)

Whence we get the first part of the theorem by observing that if:

\[
C' := \inf_{(v,w)}: \|v\| = \|w\| = 1 \ Rm(v, Jv, w, Jw) \\
\]  

then:

\[
\tilde{g}^{kl} R(g_0) \tilde{g}_{uij} \geq -C \left( \tilde{g}^{kl} g_{0kl} \right) \left( g_{0}^{ij} g_{uij} \right) = \\
- C \left( \tilde{g}^{kl} g_{0kl} - \frac{Z^k \bar{Z}^l}{|V|^2 + \epsilon} g_{uij} \right) (n + \Delta_0 u) = \\
- C \text{tr}_{g_0} (g_0)(n + \Delta_0 u) + C \frac{|V|^2}{|V|^2 + \epsilon} \geq -C \text{tr}_{g_0} (g_0)(n + \Delta_0 u) \\
\]  

and that:

\[
\hat{\Delta}_{u,e} u = \tilde{g}^{ij}_{u,e} (g_{uij} - g_{0uij}) = \left( g_{u}^{ij} - \frac{Z^i \bar{Z}^j}{|V|^2 + \epsilon} \right) g_{uij} - \tilde{g}_{u}^{ij} g_{0ij} \\
= \text{tr}_{g_0} g_{u} - \frac{Z^i \bar{Z}^j}{|V|^2 + \epsilon} g_{uij} - \tilde{g}_{e,u} g_{0ij} \\
= n - \frac{|V|^2}{|V|^2 + \epsilon} \tilde{g}_{e,u} g_{0ij}. \\
\]  

\[
\Box \\
\]  

Finally, if $|V|^2 < 1 - \epsilon$, we consider:

\[
\tilde{g}^{kl}_u := g^{kl} - \frac{(1 - \epsilon)}{|V|^2} Z^k \bar{Z}^l \\
\]  

and observe that:

\[
\tilde{g}^{kl}_u g_{0ij} \geq |V|^2_0 \ e^{-\frac{F-\lambda u}{n}} \\
\]  

(76)
Proof. The proof is the same as the one of equation (43) in Lemma 5.5. Here we merely observe that if we choose coordinates at a point such that $g_{u^j} = \lambda_i \delta_{ij}$, $Z^k = 0$ for $k < n$ and $g_{0i} = 1$:

$$\tilde{g}^{kl}_u = \lambda_k^{-1} \delta_{kl} \quad \text{for} \ k, l < n$$

and

$$\tilde{g}^{mn}_u = \frac{1}{\lambda_n} - \frac{(1 - \epsilon) - |Z^u|^2 \lambda_n}{\lambda_n} = \frac{\epsilon + |V|^2_u}{|V|^2_u + \epsilon}$$

whence $\tilde{g}^{kl}_u$ is diagonal and if $\tilde{\lambda}^i$ are its eigenvalues:

$$\prod_{i=1}^n (\tilde{\lambda}^i)^{-1} = \prod_{i=1}^{n-1} \lambda_i = \prod_{i=1}^n \lambda_i = \prod_{i=1}^n \frac{|V|^2_u}{|V|^2_u + \epsilon}$$

and since the $\epsilon$-perturbed $V$-soliton equation implies:

$$\prod_{i=1}^n \lambda_i = (|V|^2_u + \epsilon) e^{F - \lambda_u}$$

the claim follows. \qed

Theorem 5.15. If $|V|^2_u < \epsilon$, the following differential inequality holds:

$$\Delta_{u, \epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq (a - C) e^{-au} |V|^2_0 e^{-\frac{F + \lambda_u}{n}} (n + \Delta_0 u) - (a n + 2(a - C)|V|^2_0 (n + \Delta_0 u)e^{-au}$$

$$+ e^{-au}(-R(g_0) + \Delta_0 F) \right)$$

(77)

and:

$$\Delta_{u, \epsilon} \left( e^{-au} |V|^2_0 \right) \geq (a - 2C)|V|^2_0 e^{-\frac{F + \lambda_u}{n}} (|V|^2_0 e^{-au}(n + \Delta_0 u)) - (a n + 2(a - C)|V|^2_0 (|V|^2_0 (n + \Delta_0 u)e^{-au}$$

$$+ e^{-au}(-R(g_0) + \Delta_0 F) |V|^2_0 \right)$$

(78)

Proof. From equation (71) we have:

$$\Delta_{u, \epsilon} \left( e^{-au}(n + \Delta_0 u) \right) \geq e^{-au} \left( \tilde{g}^{kl}_u R(g_0)_{k'l} \tilde{g}_{u^j} - R(g_0) + \Delta_0 F \right)$$

$$- a e^{-au}(\Delta_0 u + n) \Delta_{u, \epsilon} u$$

We next notice that:

$$\tilde{g}^{kl}_u, \epsilon = g^{kl}_u - \frac{Z^k \tilde{Z}^l}{|V|^2_u + \epsilon} \tilde{g}^{kl}_u + \frac{(1 - \epsilon) - |V|^2_u}{|V|^2_u} Z^k \tilde{Z}^l - \frac{Z^k \tilde{Z}^l}{|V|^2_u + \epsilon}$$

$$= \tilde{g}^{kl}_u + \frac{\epsilon(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} Z^k \tilde{Z}^l$$

(79)
Therefore that:

\[ g_u^{kl} R(g_0)_{kl} g_{uij} \geq -C \left( \hat{g}_u^{kl} g_{0kl} \right) \left( g_0^{ij} g_{uij} \right) = \]

\[ -C \left( \hat{g}_u^{kl} g_{0kl} \right) (n + \Delta_0 u) - C|V|_0^2 \left( \frac{\epsilon(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} \right) (n + \Delta_0 u) = \]

and that:

\[ \hat{\Delta}_{u,e} u = \hat{g}_u^{kl} u_{kl} = \hat{g}_u^{kl} (g_{ukl} - g_{0kl}) = \hat{g}_u^{kl} g_{ukl} - \hat{g}_u^{kl} g_{0kl} - |V|_0^2 \left( \frac{\epsilon(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} \right) \]

whence, since \( \hat{g}_u^{kl} g_{ukl} = n - \frac{|V|^2}{|V|^2 + \epsilon} \):

\[ -\hat{\Delta}_{u,e} u = -n + \frac{|V|^2}{|V|^2 + \epsilon} + \hat{g}_u^{kl} g_{ukl} + |V|_0^2 \left( \frac{\epsilon(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} \right) \tag{80} \]

Observe that if \( |V|^2_u \leq \epsilon \) then:

\[ |V|_0^2 \left( \frac{\epsilon(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} \right) \geq |V|_0^2 \left( \frac{|V|^2_u(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} \right) = |V|_0^2 \frac{-2\epsilon}{|V|^2_u(|V|^2_u + \epsilon)} \geq -2|V|_0^2 \]

having used that (since \( \frac{1}{|V|^2 + \epsilon} \leq \frac{1}{2} \)):

\[ \frac{2\epsilon}{|V|^2 + \epsilon} = 2 \frac{\epsilon}{|V|^2 + \epsilon} \leq 2 \]

and therefore (if \( a > C \)):

\[ R(g_0)_{kl} g_{uij} - a(\Delta_0 u + n) \hat{\Delta}_{u,e} u \]

\[ \geq (a - C) \left( \hat{g}_u^{kl} g_{0kl} \right) (n + \Delta_0 u) + (a - C) |V|_0^2 \left( \frac{\epsilon(1 - \epsilon) - (1 + \epsilon)|V|^2_u}{|V|^2_u(|V|^2_u + \epsilon)} \right) (n + \Delta_0 u) \]

\[ - an(n + \Delta_0 u) \geq (a - C) \left( \hat{g}_u^{kl} g_{0kl} \right) (n + \Delta_0 u) - (an + 2(a - C)|V|^2_0) (n + \Delta_0 u) \]

Putting this equation back into eq. (71) reproduced above:

\[ \hat{\Delta}_{u,e} \left( e^{-au} (n + \Delta_0 u) \right) \geq (a - C) e^{-au} \left( \hat{g}_u^{kl} g_{0kl} \right) (n + \Delta_0 u) \]

\[ - (an + 2(a - C)|V|^2_0) (n + \Delta_0 u) e^{-au} \]

\[ + e^{-au} (-R(g_0) + \Delta_0 F) \tag{81} \]

Whence, after appealing to Lemma 5.14:

\[ \hat{\Delta}_{u,e} \left( e^{-au} (n + \Delta_0 u) \right) \geq (a - C) e^{-au} |V|^2_0 \frac{e^{-\frac{a}{n} + \lambda u}}{n + \Delta_0 u} - (an + 2(a - C)|V|^2_0) (n + \Delta_0 u) e^{-au} \]

\[ + e^{-au} (-R(g_0) + \Delta_0 F) \]
Next we show the second identity.
\[
\hat{\Delta}_{u,e} \left( |V|^{2\alpha}_0 e^{-au}(\Delta_0 u + n) \right) = |V|^{2\alpha}_0 e^{-au} \hat{\Delta}_{u,e}(\Delta_0 u + n) \\
+ \hat{\nabla}_{\hat{g}_{u,e}}(\Delta_0 u + n) \cdot \left( |V|^{2\alpha}_0 \hat{\nabla}_{\hat{g}_{u,e}}(e^{-au}) + \hat{\nabla}_{\hat{g}_{u,e}}(|V|^{2\alpha}_0) e^{-au} \right) \tag{82}
\]
\[
+ (\Delta_0 u + n) \left[ |V|^{2\alpha}_0 e^{-au} \left( -a \hat{\Delta}_{u,e} u + a^2 |\nabla_{\hat{g}_{u,e}} u|^2 \hat{g}_{u,e} \right) + e^{-au} \hat{\Delta}_{u,e} |V|^{2\alpha}_0 \right]
\]
Direct computations yield:
\[
\hat{\nabla}_{u,e} |V|^{2\alpha}_0 = \alpha |V|^{2(\alpha-1)}_0 \hat{\nabla}_{u,e} |V|^{2}_0 \tag{83}
\]
and:
\[
\hat{\Delta}_{u,e}( |V|^{2\alpha}_0 ) = \alpha |V|^{2(\alpha-1)}_0 \hat{\Delta}_{u,e} |V|^{2}_0 + \alpha (\alpha - 1) |V|^{2(\alpha-2)}_0 \left| \hat{\nabla}_{u,e} |V|^{2}_0 \right|
\]
Next observe that by Cauchy-Schwartz one has:
\[
-a |V|^{2\alpha}_0 \hat{\nabla}_{\hat{g}_{u,e}}(\Delta_0 u) \cdot \hat{\nabla}_{\hat{g}_{u,e}}(u) + \alpha |V|^{2(\alpha-1)}_0 \hat{\nabla}_{\hat{g}_{u,e}}(\Delta_0 u) \cdot \nabla_{\hat{g}_{u,e}}(|V|^{2}_0) \geq -a |V|^{2\alpha}_0 \hat{\nabla}_{\hat{g}_{u,e}}(\Delta_0 u) \cdot \hat{\nabla}_{\hat{g}_{u,e}}(u) - \alpha |V|^{2(\alpha-1)}_0 |\hat{\nabla}_{\hat{g}_{u,e}}(\Delta_0 u) \cdot \nabla_{\hat{g}_{u,e}}(|V|^{2}_0)| \geq - |V|^{2\alpha}_0 \left( \frac{|\nabla_{\hat{g}_{u,e}}(\Delta_0 u)|^2}{\Delta_0 u + n} + a^2 |\nabla_{\hat{g}_{u,e}} u|^2_{\hat{g}_{u,e}}(\Delta_0 u + n) \right) \tag{84}
\]
Therefore, going back to (82) (and using (83)):
\[
\hat{\Delta}_{u,e} \left( |V|^{2\alpha}_0 e^{-au}(\Delta^H_0 u + m) \right) \geq |V|^{2\alpha}_0 e^{-au} \hat{\Delta}_{u,e}(\Delta^H_0 u + m) \\
-2 |V|^{2\alpha}_0 \left| \nabla_{\hat{g}_{u,e}}(\Delta^H_0 u) \right|^2_{\hat{g}_{u,e}} - a |V|^{2\alpha}_0 e^{-au} \hat{\Delta}_{u,e} u \]
\[
+ \alpha e^{-au}(\Delta^H_0 u + m) |V|^{2(\alpha-1)}_0 \left( \hat{\Delta}_{u,e} |V|^{2}_0 - \frac{|\hat{\nabla}_{u,e} |V|^{2}_0|^2}{|V|^{2}_0} \right) \tag{85}
\]
Therefore, appealing to equation (81):
\[
\hat{\Delta}_{u,e} \left( |V|^{2\alpha}_0 e^{-au}(\Delta^H_0 u + m) \right) \geq |V|^{2\alpha}_0 (a - C) e^{-au} \left( \hat{g}_{u,e}^{k\bar{k}} g_{0k\bar{k}} \right) (n + \Delta_0 u) \\
-(an + 2(a - C)|V|^{2\alpha}_0 |V|^{2\alpha}_0 (n + \Delta_0 u)e^{-au} \\
+ e^{-au}|V|^{2\alpha}_0 (-R(g_{0}) + \Delta_0 F) \]
\[
+ \alpha e^{-au}(\Delta^H_0 u + m) |V|^{2(\alpha-1)}_0 \left( \hat{\Delta}_{u,e} |V|^{2}_0 - \frac{|\hat{\nabla}_{u,e} |V|^{2}_0|^2}{|V|^{2}_0} \right) \tag{85}
\]
Next recall equation (53) in Lemma 5.7, which we can rewrite as:

\[ \hat{\Delta}_{u,\epsilon}(|V|_0^2) - \frac{|\nabla_{u,\epsilon}|V|_0^2|^2}{|V|_0^2} \geq -C_0|V|_0^2 (\hat{g}^{ij}_u g_{0ij}) \]  

(86)

where \( C_0 := \sup_{i,k} g^{ij}_0 g^{ij}_0 R(g_0) \hat{g}^{kl} Z^k Z^l |V|_0^2 \) (which we may assume, without loss of generality, to be the same as the constant \( C \) in eq. (52)).

\[
\hat{g}^{kl}_{u,\epsilon} = g^{kl}_u - \frac{Z^k Z^l}{|V|_0^2 + \epsilon} \geq g^{kl}_u + \frac{(1 - \epsilon) - |V|_0^2}{|V|_0^2 + \epsilon} Z^k Z^l - \frac{Z^k Z^l}{|V|_0^2 + \epsilon} \]

(87)

Using equation (87) (and that \( |V|_0^2 \leq \epsilon \)) this becomes:

\[
\hat{\Delta}_{u,\epsilon}(|V|_0^2) - \frac{|\nabla_{u,\epsilon}|V|_0^2|^2}{|V|_0^2} \geq -C \epsilon^2 \hat{g}^{kl}_u g_{0ij} \]

(88)

\[
\hat{\Delta}_{u,\epsilon}(|V|_0^2) - \frac{|\nabla_{u,\epsilon}|V|_0^2|^2}{|V|_0^2} \geq -C \epsilon^2 \frac{(1 - \epsilon) - (1 + \epsilon) |V|_0^2}{|V|_0^2 + \epsilon} \]

and therefore:

\[
|V|_0^{2a} R(g_0) \hat{g}^{ij}_{0,kl} g^{ij}_{0,kl} - a e^{-au} (\Delta_0 u + n) \hat{\Delta}_{u,\epsilon} u + \alpha (\Delta_0 H u + m) |V|_0^{2(a-1)} \left( \hat{\Delta}_{u,\epsilon} |V|_0^2 - \frac{|\nabla_{u,\epsilon}|V|_0^2|^2}{|V|_0^2} \right) \]

\[
\geq (a - C) |V|_0^{2a} \left( \hat{g}^{kl}_u g_{0kl} \right) (n + \Delta_0 u) + (a - C) |V|_0^{2a} |V|_0^{2a} \left( \frac{(1 - \epsilon) - (1 + \epsilon) |V|_0^2}{|V|_0^2 + \epsilon} \right) (n + \Delta_0 u) \]

\[
- a |V|_0^{2a} n(n + \Delta_0 u) - C |V|_0^{2a} \hat{g}^{kl}_u g_{0kl} (n + \Delta_0 u) - C (n + \Delta_0 u) \frac{(1 - \epsilon) - (1 + \epsilon) |V|_0^2}{|V|_0^2 + \epsilon} (n + \Delta_0 u) \]

\[
\geq \left( (a - 2C) \hat{g}^{kl}_u g_{0kl} \right) (n + \Delta_0 u) - (a + 2(a - 2C) |V|_0^2) (n + \Delta_0 u) \]

\[
|V|_0^{2a} \]

so choosing \( a > 2C \) and going back to equation (85)

\[
\hat{\Delta}_{u,\epsilon} |V|_0^{2a} e^{-au} (\Delta_0 u + m) \geq |V|_0^{2a} (a - 2C) e^{-au} \hat{g}^{ij}_u g_{0ij} \]

(89)

\[
- (a + 2(a - 2C) |V|_0^2) (n + \Delta_0 u) e^{-au} \]

\[
+ e^{-au} |V|_0^{2a} (-R(g_0) + \Delta_0 F) \]

having used the fact that \( \hat{g}^{ij}_u g_{0ij} = g^{ij}_u g_{0ij} - \frac{Z^k Z^l}{|V|_0^2} g^{ij}_u g_{0ij} = g^{ij}_u g_{0ij} - \frac{|V|_0^2}{|V|_0^2} \).

We finally make use of eq. (53) from Lemma 5.7 and equation (72) in Lemma 5.12 to conclude the proof.

□
6. The $C^2$ Estimate

In this section, we give an a priori $C^2$-estimate on solutions of (8). For simplicity, we write $F$ for $F_{c,s}$. Let $u$ be a bounded solution of (8).

Consider a unitary frame $e_1, \ldots, e_n$ such that $e_n = \frac{Z}{|V|_0}$. The one can easily prove:

**Lemma 6.1.** $||\bar{\partial} \partial u(e_i, \bar{e}_j)||_{L^\infty} \leq \max\{2 \max\{||J\nabla u||_{L^\infty}, m + \hat{\Delta}_h u\}, m\}$

**Proof.** This follows from the fact (cf. equation (10)) that the horizontal projection (onto $Q^{(1,0)}_0$ $\wedge$ $Q^{(0,1)}_0$) of the complex Hessian is:

$$[\partial \bar{\partial} \phi]_h = \sum \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j + \frac{1}{4} \left( J\nabla(\phi) \frac{\partial h_{ij}}{\partial \tau} \right) dz_i \wedge d\bar{z}_j$$

Therefore, in order to derive a priori horizontal $C^2$-estimates, we only need to have a $C^0$-estimate for $\Delta g_0 u$. Also, we have:

**Lemma 6.2.** Let $u$ be a solution of the $\epsilon$-perturbed $V$-soliton equation:

$$\omega^n = (|V|^2_u + \epsilon) e^{F-a\omega_0}$$

Then $\omega_u > 0$ for any $\epsilon > 0$ and in particular $|V|^2_u(p) > 0$ for any $\epsilon > 0$ and any $p \in M \setminus Z$ where $Z := \{ x \in M : V(p) = 0 \}$. Moreover, if $|V|^2_u > \kappa$, then:

$$\omega^n_u > C(\kappa, ||F||_{L^\infty}, ||u||_{L^\infty}) \omega^n_0$$

where $C(\kappa, ||F||_{L^\infty}, ||u||_{L^\infty}) = \kappa \inf_M (e^{F-a\omega})$

**Proof.** Straightforward. 

We are now ready to prove the main theorem.

**Theorem 6.3.** There exists a constant $C$, depending only on $(M, g_0)$, $V$ and $\max\{||F||_{L^\infty}, ||\Delta^H_0 F||_{L^\infty}\}$ and $||u||_{L^\infty(M)}$, such that, for any $x \in M$:

$$0 < (n - 1 + \hat{\Delta}_0^H u)(x) < C.$$  \hspace{1cm} (90)

In particular:

$$||\hat{\nabla}^H \hat{\nabla}^H u||_{L^\infty} < C.$$

**Proof.** Let $\epsilon_i$ be a sequence of number such that $\lim_{i \to \infty} \epsilon_i = \epsilon_0 \geq 0$ and assume that for each $i$ we have a solution to the $\epsilon_i$-perturbed $V$-soliton equation. From now on let us set:

$$y_{H,i} := e^{-au_i}(n - 1 + \hat{\Delta}_0^H u_i) \text{ and } y_i := e^{-au_i}(n + \Delta_0 u_i)$$

Remark that:

$$y_i = y_{H,i} + e^{-au_i} \frac{|V|^2_{u_i}}{|V|^2_{0}}$$
By an easy application of the maximum principle to eq. (69) in Theorem 5.11, we have immediately that if $|V|^2_{u_i} \leq A$ for some number $A$, then at a max $p_i$ of $y_i$ one has:

$$c_1 y_i^{\frac{n}{n-1}} \leq (|V|^2_{u_i} + \epsilon)^{\frac{n}{n-1}} (a(n+1)y + c_3) \leq A^{\frac{n}{n-1}} (a(n+1)y + c_3)$$

where $c_3 := \|e^{-au}\|_{\infty} (\|R(g_0)\|_{\infty} + \|\Delta_0 F\|_{\infty})$. Thus $y_i$ is uniformly bounded (and equivalently $y_{H,i}$ as, since $|V|^2_{u_i} \leq A$):

$$\sup y_i \leq \max \left\{ \frac{c_3}{a(n+1)}, A \frac{a(n+1)}{c_3} \right\}$$

(91)

So we may assume that $\lim_{i \to \infty} |V|^2_{u_i} = +\infty$. Given any real number $\rho > 0$, set:

$$M_{\rho,i} := \{ x \in M : |V|^2_{u_i} \geq \rho \}$$

Let $p_i$ be a point where the maximum of $y_{H,i} = e^{-au_i}(m + \Delta_0^H u_i)$ is achieved. Given a real number $\kappa$, there are two possibilities:

1. either there exists a subsequence $\{j_i\}$ such that $p_{j_i} \in M \setminus M_{\kappa,i}$ for every $\kappa > 0$
2. or there exists an index $i_0$ such that $p_i \in M_{\kappa,i}$ for every $i \geq i_0$ for some $\kappa > 0$

We are going to show that either way we can bound $\sup_M y_{H,i}$ uniformly. In case item (1) above holds, let us abuse notation and indicate $u_{j_i}$ with $u_i$. Then by assumption $|V|^2_{u_i}(p_i) \leq \kappa$.

Let $q_i$ be points such that $\sup_M e^{-au_i}(n + \Delta_0 u_i) = e^{-au_i}(n + \Delta_0 u_i)(q_i)$. There are two possibilities:

- either there exists a subsequence $\{j_i\}$ such that $q_{j_i} \in M \setminus M_{\kappa}$
- or there exists an index $i_0$ such that $q_i \in M_{\kappa}$ for every $i \geq i_0$

So either way, reformulating, we have two possibilities in case (1) for the the sequences $\{p_i\}$ and $\{q_i\}$ (in the case in which $p_i \in M \setminus M_{\kappa}$):

- either $p_i \in M \setminus M_{\kappa}$ and $q_i \in M_{\kappa}$
- or $p_i \in M \setminus M_{\kappa}$ and $q_i \in M \setminus M_{\kappa}$ for any $\kappa$.

In the first case we appeal to Lemma 6.4 below to conclude that there exists a $K > 0$ such that $\sup_M |V|^2_{u_i} \leq K$ and therefore, using equation (69) from Theorem 5.11 coupled with Lemma 5.12,

$$\hat{\Delta}_{u_i} \left( e^{-au}(n + \Delta_0 u) \right) \geq (a - C) \frac{e^{-\frac{E_{\Delta u}}{n-1}}}{(|V|^2_{u_i} + \epsilon)^{\frac{1}{n-1}}} \left( e^{-au}(n + \Delta_0 u) \right)^{\frac{n}{n-1}} \text{tr}_{g_u}(g_0)$$

$$- a \left( n - \frac{|V|^2_{u_i}}{|V|^2_{u_i} + \epsilon} \right) \frac{|V|^2_{u_i}}{|V|^2_{u_i} + \epsilon} \left( e^{-au}(n + \Lambda_0 u) \right)$$

$$- \|e^{-au}\|_{\infty} (\|R(g_0)\|_{\infty} + \|\Delta_0 F\|_{\infty})$$
where \( C := \inf_{(v,w): \|v\| = \|w\| = 1} Rm(v, Jv, w, Jw) \). Therefore at the max \( q_i \):

\[
\sup_M e^{-au_i}(n + \Delta_0 u_i) \leq \max \left\{ 2 \frac{c_4^{n-1}}{c_5} (K + \epsilon_i), c_3 \right\}
\]

where:

\[
c_3 := \|e^{-au}\|_\infty (\|R(g_0)\|_\infty + \|\Delta_0 F\|_\infty),
\]

\[
c_4 := a \left( n - \frac{\|V_{0i}^2\|}{\kappa + \epsilon_i} \right)
\]

and

\[
c_5 := (a - C)\|e^{-\frac{F - \lambda u}{\kappa - 1}}\|_\infty
\]

In the second sub case of case (1) we further have to distinguish between the following:

- either \( |V_{0i}^2(q_i)| \geq \epsilon_i \)
- or \( |V_{0i}^2(q_i)| < \epsilon_i \)

If the first possibility occurs, we can appeal to equation (74) in Theorem 5.13 which implies that the point \( q_i \) which is t a max for \( e^{-au}(n + \Delta_0 u_i) \)

\[
(a - C) e^{-au}(n + \Delta_0 u) \left( \frac{\|V_{0i}^2\| |V_{0i}^2|}{(\|V_{0i}^2\| + \epsilon) \det(g_0)} \right)^{\frac{1}{n-1}} e^{-\frac{F_\Lambda}{\kappa - 1}} (m + \Delta_0^H u) \frac{1}{n-1}
\]

, since \( n + \Delta_0 u_i = (m + \Delta_0^H u_i) + \frac{|V_{0i}^2|}{|V_{0i}^2|} \):

\[
\sup_M e^{-au_i}(m + \Delta_0^H u_i) \leq \sup_M e^{-au_i}(n + \Delta_0 u_i) \leq \max \left\{ 2 \frac{c_4^{n-1}}{c_5} (K + \epsilon_i), c_3 \right\}
\]

and we are done with this sub case of case (1). So, still in case (1), the other possibility is that \( p_i \in M \setminus M_\kappa \) and \( q_i \in M \setminus M_\kappa \) for any \( \kappa \). In this case:

\[
e^{-au_i(p_i)}(m + \Delta_0^H u_i)(p_i) \leq e^{-au_i(q_i)}(n + \Delta_0 u_i)(q_i) = e^{-au_i(q_i)}(m + \Delta_0^H u_i)(q_i) + e^{-au_i(q_i)}\frac{|V_{0i}^2|}{|V_{0i}^2|} (q_i)
\]

and we can therefore bound \( \sup_M e^{-au_i}(m + \Delta_0^H u_i) \) by bounding \( e^{-au_i(q_i)}(m + \Delta_0^H u_i)(q_i) \). We achieve this by applying the maximum principle to equation (78) in Theorem ??, which says that if \( |V_{0i}^2| \leq \epsilon \):

\[
\hat{\Delta}_{u, \epsilon} (e^{-au}|V_{0i}^{2 \alpha}(n + \Delta_0 u))
\]

\[
\geq (a - 2C)|V_{0i}^{2 \alpha} e^{-\frac{F - \lambda u}{n - 1}} (|V_{0i}^{2 \alpha} e^{-au}(n + \Delta_0 u) |(\|V_{0i}^{2 \alpha} e^{-au}(n + \Delta_0 u) \|_\infty - \epsilon) - (an + 2(a - C)|V_{0i}^{2 \alpha} (|V_{0i}^{2 \alpha} (n + \Delta_0 u)e^{-au})
\]

+ e^{-au} (-R(g_0) + \Delta_0 F) |V_{0i}^{2 \alpha}
\]
Thus at a max for \( e^{-au}|V|_{0}^{2a} (n + \Delta_0 u) \)

\[
(a - 2C)|V|_{0}^{2} e^{-\frac{R_{b,\lambda}}{a}} \left( |V|_{0}^{2a} e^{-au} (n + \Delta_0 u) \right)^{\frac{\alpha}{n + 1}} 
\leq (an + 2(a - C)|V|_{0}^{2}) \left( |V|_{0}^{2a} (n + \Delta_0 u) e^{-au} \right) 
- \|e^{-au} (-R(g_0) + \Delta_0 F) \|_{\infty} |V|_{0}^{2a} 
\]

If on the other hand case (2) were to hold, then from equation (56) we know that:

\[
\tilde{\Delta}_{\epsilon,u} \left( e^{-au}(\Delta_0^H u + m) \right) - 2A\beta JV \left( e^{-au}(\Delta_0^H u + m) \right) \geq 
\]

\[
c_1 e^{-au} (m + \Delta_0^H u)^{\frac{m}{m + 1}} - \left( c_2 + \frac{c'_2}{|V|_{0}^{2}} \right) - (c_3 + \beta)e^{-au}(m + \Delta_0^H u). 
\tag{92}
\]

where \( c_1 := (a - C) \inf_{M} e^{-\frac{\kappa}{n - 1}} > 0, c_2 := a \sup_{M} e^{-au} |\Delta_0 F| \) and \( c_3 := a \left( n - \frac{|V|_{0}^{2}}{|V|_{0}^{2} + 1} \right) \).

Whence, since \( x_i \in \mathcal{M}_\kappa \):

\[
\beta(x_i) = \frac{\epsilon}{|V|_{0}^{2}(x_i)(|V|_{0}^{2}(x_i) + \epsilon)} \leq \frac{\epsilon}{\kappa(\kappa + \epsilon)}
\]

it follows by an application of the maximum principle to equation (68) that:

\[
c_1 e^{-au} (m + \Delta_0^H u)^{\frac{m}{m + 1}} \leq \left( c_2 + \frac{c'_2}{|V|_{0}^{2}} \right) + (c_3 + \beta)e^{-au}(m + \Delta_0^H u) 
\leq \left( c_2 + \frac{c'_2}{|V|_{0}^{2}} \right) + \left( c_3 + \frac{\epsilon}{\kappa(\kappa + \epsilon)} \right) e^{-au}(m + \Delta_0^H u)
\]

and once again either \( y_{H,i} \) is uniformly bounded or the sup of \( y_{H,i} \) is at the boundary of \( \mathcal{M}_\kappa \), but the latter can be avoided, by making \( \kappa \) slightly larger. In fact, if \( x_i \) accumulated towards the boundary of \( \mathcal{M}_\kappa \), i.e., \( \partial \mathcal{M}_\kappa = \{ x \in M : |V|_{0}^{2}(x) = \kappa \} \), then by taking a larger \( \kappa \), say \( \kappa' \), there would be an index \( i_0 \) such that \( x_i \in M \setminus \mathcal{M}_{\kappa'} \) for every \( i \geq i_0 \) and we would be in the case dealt with before (i.e., case (1)).

Next we prove the final piece needed for the proof of the theorem above, which we postponed till now as mentioned during the proof of the mentioned theorem:

**Lemma 6.4.** Let \( \epsilon_i > 0 \) a convergent sequence and let \( u_i \) be a solution to:

\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} u_i) = (|V|_{0}^{2} + \epsilon_i) e^{F - \lambda u_i} \omega_0^n 
\]

Let \( p_i \in M \) be a point such that \( e^{-au_i(p_i)}(n + \Delta_0 u_i)(p_i) = \sup_{M} e^{-au}(n + \Delta_0 u_i) \) and assume that there exists \( \kappa > 0 \) such that \( |V|_{u_i}^{2} \geq \kappa \). Then there exists a constant \( C > 0 \) and a radius \( r > 0 \) such that:

\[
\sup_{B_{r}(p)} |V|_{u_i}^{2} \leq C
\]

for every \( i \).
Proof. The proof is by contradiction and by blow-up analysis and is similar in spirit to the proof of Theorem 2 in [X.X. Chen]. So we assume that there exists a sequence of counterexamples to the Lemma and derive a contradiction. Thus, let \{u_i\} be a (sub)sequence such that:

\[ \mu_i^2 := \sup_{B_r(p_i)} |V|_{u_i}^2 \to +\infty \]

for every \( r > 0 \). Since \( |V|_{u_i}^2 = |V|_0^2 + Z\bar{Z}(u_i) \) we may assume that:

\[ \sup_{B_r(p)} Z\bar{Z}(u_i) \to +\infty \]

and that we have taken \( i \) so large that:

\[ \lambda_i^2 := \sup_{B_r(p)} Z\bar{Z}(u_i) > 0. \]

Let \( x_i \in \overline{B}_r(p) \) such that \( \lambda_i^2 = Z\bar{Z}(u_i)(x_i) \). By compactness of \( M \), up to subsequences, we may assume \( \{x_i\} \) converges to \( x \in M \). Let \( \delta > 0 \) sufficiently small such that the background metric is nearly Euclidean in \( B_\delta(x) \) and such that \( B_\delta(x) \subset B_{2\delta}(p) \).

Let \( B_{\lambda_i\delta} \subset \mathbb{C}^n \) be the Euclidean ball centered at the origin of radius \( \lambda_i \delta \). Consider the holomorphic map:

\[ \Psi_i : B_{\lambda_i\delta} \to B_\delta(x_i) \]

given by:

\[ \Psi_i(z) = x_i + \frac{z}{\lambda_i} \]

and pull-back all the (rescaled) geometric quantities by it. In particular, we set:

\[ v_i(x) := u_i(x_i + \frac{x}{\lambda_i}), \]

\[ G_0 := \Psi_i^*(\lambda_i^2 g_0) = g_{0kl}(x_i + \frac{x}{\lambda_i}) dz_k \wedge d\bar{z}_l, \]

\[ G_{u_i} = \Psi_i^*(\lambda_i^2 g_{u_i}) = g_{ukl}(x_i + \frac{x}{\lambda_i}) dz_k \wedge d\bar{z}_l \]

and:

\[ W := \Psi_i^*(\lambda_i Z) = \sum_k Z^k \frac{\partial}{\partial z_k} \quad \hat{Z} := \Psi_i^* Z \]

where \( Z^k \) are the components of \( Z = \sum_k Z^k \frac{\partial}{\partial w_k} \). Note that:

\[ \Delta_{G_{u_i}} v_i = (\Delta_{g_{u_i}}} u_i) \circ \Psi_i \]

and since by assumption \( Z\bar{Z}(x_i) = \lambda_i^2 - |V|_0^2 \), one gets that:

\[ W\hat{W}(v_i) = \frac{1}{\lambda_i^2} \left( Z^k \hat{Z}^l \frac{\partial^2}{\partial w_k \partial \bar{w}_l} \right) \left( x_i + \frac{x}{\lambda_i} \right) = 1 - \frac{|V|_0^2}{\lambda_i} \]
By equation (70) coupled with Lemma 5.12 one has:

\[ \Delta_{u,e} \left( e^{-au} (n + \Delta_0 u) \right) \geq (a - C) \left( e^{-au} (n + \Delta_0 u) \right) \frac{e^{F_{\lambda} u}}{\left( |V|^2 + \epsilon \right)^{\frac{1}{n-1}}} \left( e^{-au} (n + \Delta_0 u) \right)^{\frac{1}{n-1}} \]

(93)

\[ - c_2 e^{-au} (n + \Delta_0 u) - \|e^{-au}\|_\infty (\|R(g_0)\|_\infty + \|\Delta_0 F\|_\infty) \]

where:

\[ c_2 = c_2(a, n, |V|^2, |V|_{\alpha}^2) := n + \frac{|V|^2}{|V|_{\alpha}^2 + \epsilon} > 0 \]

and since \( |V|^2 (p) \geq \kappa \), it follows that at the max \( p_i \):

\[ c_1 \left( e^{-au} (n + \Delta_0 u) \right)^{\frac{n}{n-1}} \leq (|V|^2(p_i) + \epsilon)^{\frac{1}{n-1}} c_2' e^{-au} (n + \Delta_0 u) + c_3 \]

(94)

with

\[ c_2' = n + \frac{|V|^2}{\kappa + \epsilon} \]

and:

\[ c_3 := \|e^{-au}\|_\infty (\|R(g_0)\|_\infty + \|\Delta_0 F\|_\infty) \]

which implies that either \( e^{-au} (n + \Delta_0 u) \leq c_3 \) (in which case \( n + \Delta u_i = m + \Delta_0 H u_i + |V|^2 |V|_{\alpha}^2 \leq \|e^{au}\|_\infty c_3 \) hence \( |V|^2 \leq c_3 \|e^{au}\|_\infty \|V|^2 \)) or:

\[ (e^{-au} (n + \Delta_0 u) \leq \left( \frac{c_2'}{c_1} \right)^{n-1} (\lambda_i + \epsilon) \]

whence there exists a uniform constant \( K \):

\[ 0 < \frac{1}{\lambda_i} g_{0kl} + \sqrt{-1} \frac{\partial^2 u_i}{\partial w_k \partial \bar{w}_l} < K (\lambda_i + \epsilon) \]

where the lower bound stems from the fact that we are in the region where \( |V|^2 \geq \kappa \) (so we can apply Lemma 6.2).

Therefore, \( v_i \) converges strongly in \( C^{1,\alpha} \) for every \( \alpha \) and weakly \( W_{loc}^{2,p} \) for every \( p > 1 \) to a plursubharmonic bounded function \( v_\infty \). Since the convergence is strongly in \( C^{1,\alpha} \), we can restrict the limiting function \( v_\infty \) to any complex line \( L \in G(2, n) \) and obtain a bounded weakly sub-harmonic function on \( L \). Since any bounded weakly sub-harmonic function on the two dimensional plane is constant, we must have that \( v_\infty \) is thus constant, which is in contradiction with \( \text{W} \overline{W} (v_\infty) = 1 \).

Therefore, \( v_i \) converges strongly in \( C^{1,\alpha} \) for every \( \alpha \) and weakly \( W_{loc}^{2,p} \) for every \( p > 1 \) to a plursubharmonic bounded function \( v_\infty \). Since the convergence is strongly in \( C^{1,\alpha} \), we can restrict the limiting function \( v_\infty \) to any complex line \( L \in G(2, n) \) and obtain a bounded weakly sub-harmonic function on \( L \). Since any bounded weakly sub-harmonic function on the two dimensional plane is constant, we must have that \( v_\infty \) is thus constant, which is in contradiction with \( \text{W} \overline{W} (v_\infty) = 1 \).

\[ \square \]

6.1. **The full a priori \( C^2 \)-estimate.** Recall we denote \( \mathcal{Z} := \{ x \in M : V(x) = 0 \} \). Here we prove:

**Theorem 6.5.** There exists a constant \( C \), depending only on \( (M, g_0) \), \( V \) and \( \|F\| \) and \( \|u\|_{L^\infty(M)} \), such that, for any \( x \in M \):

\[ 0 < |V|^2 (n + \Delta_0 u)(x) < C. \]

(95)
In particular, for any compact set $K \subset M \setminus Z$, there exists a $C = C(K, \kappa, v, \|F\|_{C^2})$ depending only on the compact set $K$, $\|F\|_{C^2}$, $\kappa := \|\nabla H \nabla H u\|_{L^\infty}$ and $v := \|\frac{1}{\sqrt{g}}\|_{C^2}$ such that:

$$\|\nabla \nabla u\|_{L^\infty} < C.$$

**Proof.** This is an immediate consequence of Theorem 6.3 and of equation (26) in Theorem 4.2, which in particular implies (in conjunction with Moser’s parabolic sup estimate) that $\sup_{B_r(p)} |V|^2_u \leq C \int_M |V|^2_u$ for any ball $B_r(p) \subset \mu^{-1}(I)$ for $I = [-\alpha, -\delta)$ or $I = (\delta, \Omega]$ and thus, since by the argument in Theorem 6.3 we also have that $|V|^2_u \leq C$ in $M_\delta$, we have a uniform estimate for $|V|^2_u$ on $M$. This implies that:

$$n + \Delta_0 u = n - 1 + \Delta_0 H u + \frac{|V|^2_u}{|V|^2_0} \leq C + \frac{C'}{|V|^2_0}$$

\[\square\]

**References**


[La Nave-Tian10] Scalar $V$-soliton equation II: manifolds with boundary


