An Introduction to the Poincaré Polyhedron Theorem

(staying in 2D for simplicity)

- \( X \) simply connected metric space (for us \( \mathbb{R}^2 \) \( \mathbb{H}^2 \))
- \( P \) convex, finite-sided polygon in \( X \)
- \( S \) side of \( P \) (i.e. a maximal convex subset of \( \partial P \))
- We call \( g \in Isom^+(X) \) a **side-pairing** if \( \exists \) a side \( S \) s.t. \( g(S) = S' \) is a side of \( P \). Can write \( g = g_S \). Note that \( g^* = g_{S'} \) is also a side-pairing.
- \( v \in \partial P \) is a vertex if \( \exists v = S_1, S_2, S_3 \) distinct sides.

Given a set of side-pairings \( \Xi \) for \( P \), a **vertex cycle** is an equivalence class of vertices under \( \Xi \) (here we include points of \( \partial P \) fixed by elements of \( \Xi \)).

We can write a vertex cycle (sometimes just called a cycle) as a word in \( \Xi \)

\[
\begin{align*}
\text{eg} \quad X &= \mathbb{R}^2 \quad \Xi = \{ A, B, A, B \} \\
A(x,y) &= (x+1, y) \\
B(x,y) &= (x, y+1)
\end{align*}
\]

One vertex cycle:

Start with \((0,0)\); \( A \) sends to \((1,0) \), \( B \) to \((1,1) \), \( A' \) to \((0,1) \), \( B' \) to \((0,0) \)

So here \( W = B'A'BA \).
Given \( \gamma \in \text{Isom}^+(\mathbb{H}) \) discrete, \( P \) is a fundamental domain if

\[
\begin{align*}
(1) & \; \forall \; g \neq 1 \; \exists \; \gamma \gamma g(\gamma) = \emptyset \; \text{and} \\
(2) & \; \bigcup_{g \in \Gamma} g(P) = \mathbb{H}.
\end{align*}
\]

Poincare's Theorem

Let \( P, \Gamma \) be as above. Suppose that for each vertex cycle, the sum of the internal angles is \( \frac{2\pi}{n} \), some \( n \in \mathbb{N} \). Then the group \( \Gamma \) generated by \( \Gamma \) is discrete, and \( P \) is a fundamental domain for \( \Gamma \). Furthermore, we have a presentation

\[
\Gamma = \langle \Gamma | \forall \text{cycle } \gamma \gamma, \gamma^n = 1 \rangle.
\]

\[\begin{array}{c}
\text{angle sum} = 4 \left( \frac{\pi}{2} \right) = 2\pi, \; \text{so} \; n = 1
\end{array}\]

\[\therefore \; \Gamma = \langle A, B | B A^{-1} B A = 1 \rangle \cong \mathbb{Z}^2.\]
Remark 1: When writing $w$, we have a choice of direction to start.

If we instead started with $B$ in our example, we would produce $w = A'B'B'A$.

But $A'B'B'A = 1 \iff (A'B'B'A)^2 = 1 \iff B'A'B'A = 1$.

Switching direction produces the inverse relation $w' \neq 1$, which is equivalent to the original $w = 1$. So this choice does not affect the result.

Remark 2: We also have a choice of starting point.

Try starting here.

$AB'A'B = 1 \iff A'(AB'A'B)A = A'A = 1 \iff B'A'B'A = 1$

So a different starting point produces a word conjugate to the original $w$, and hence the same relation.
Q: Why these relations?

If \( G \) is to be discrete, the images of \( P \) must "fit together nicely" around each vertex. This only happens if the angle sum is \( \frac{2\pi}{n} \).

\[ \text{eg around } (0,0) \quad A'(p) \quad \text{Images of } P \text{ fit nicely} \quad \text{around } (0,0) \]

\[ B'(p) \quad B'^{-1}(p) \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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More examples:

\[ \mathbb{H}^2 \]

- A regular octagon, each angle \( \frac{\pi}{4} \).

- \( W = D'C'D'C'B'A'B'A \)

- Angle sum = \( 8 \left( \frac{\pi}{4} \right) = 2\pi \)

- So \( G = \langle A, B, C, D \mid [CD][AB] = 1 \rangle \)

- \( \cong \pi_1(\Sigma_2) \), \( \Sigma_2 \) closed genus-2 surface.

- \( PSL_2(\mathbb{Z}) \)

- \( G = \langle T, R \mid R^2 = (TR)^3 = 1 \rangle \)
  
  \( (T^\infty = 1) \)

This example raises the following question:
Q: What about vertices at \( \infty \)?

- If the ideal vertex \( v_\infty \) is fixed by the side-pairing (i.e., \( w = T \), the word has length 1) then informally we may treat \( v_\infty \) as a vertex cycle with angle sum 0 ("\( \frac{2\pi}{\infty} \"), so the pairing \( T \) has infinite order.

- If the ideal vertex is not fixed, we need a technical condition on the cycle (or rather, it would be nice for when we consider the quotient space) but we can extend in a similar way.

\[
\text{Example: } \quad G = \langle A, B \mid - (B^*A^*BA)^{\infty} = 1 \rangle
\]
One way I apply the Poincaré Polyhedron Theorem

Consider \( \Gamma_0(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) \mid c \equiv 0 \mod 2 \} \).

We would like a presentation so that we may look at properties of this group.

\( \Gamma_0(n) \subset \text{PSL}_2(\mathbb{Z}) \) so we know it is discrete.

Claim that the following is a fundamental domain: (for action on upper half-space model for \( \mathbb{H}^2 \))

2 ideal vertices
9 vertices \( \Rightarrow \) 3 cycles

\( B^{-1}DC = 1 \Rightarrow B = DC \)
\( E^{-1}FC = 1 \Rightarrow E = FC \)
\( F^{-1}DE = 1 \Rightarrow F^{-1}DFC = 1 \)

\( \Rightarrow C = F^{-1}DF \)

So \( G = \langle A, D, F \rangle \) is free of rank 3.

This fits with the fact that \( \mathbb{H}^2/\Gamma_0(n) \) is a twice-punctured torus.